

Off-Shell Quantum Electrodynamics

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Abstract

More than twenty years have passed since the threads of the ‘proper time formalism’ in covariant classical and quantum mechanics were brought together to construct a canonical formalism for the relativistic mechanics of many particles. Drawing on the work of Fock, Stueckelberg, Nambu, Schwinger, and Feynman, the formalism was raised from the status of a purely formal mathematical technique to a covariant evolution theory for interacting particles. In the context of this theory, solutions have been found for the relativistic bound state problem, classical and quantum scattering in relativistic potentials, as well as applications in statistical mechanics.

It has been shown that a generalization of the Maxwell theory is required in order that the electromagnetic interaction be well-posed in the theory. The resulting theory of electromagnetism involves a fifth gauge field introduced to compensate for the dependence of the gauge transformation on the invariant time parameter; permitting such dependence relaxes the requirement that individual particles be on fixed mass shells and allows exchange of mass during scattering. In this paper, we develop the quantum field theory of off-shell electromagnetism, and use it to calculate certain elementary processes, including Compton scattering and Møller scattering. These calculations lead to *qualitative* deviations from the usual scattering cross-sections, which are, however, small effects, but may be visible at small angles near the forward direction. The familiar IR divergence of the Møller scattering is, moreover, completely regularized.

1 Introduction

Covariant Quantum Mechanics

In 1941, Stueckelberg [1] (see also Fock [2]) proposed a Poincaré invariant Hamiltonian mechanics in which particle worldlines are traced out by the evolution of events according to an invariant parameter τ . Stueckelberg's purpose was to describe pair annihilation as the evolution of a single worldline, first forward and then backward in time. Since the Einstein time coordinate $x^0 = t$ does not increase monotonically as the system evolves in such a model (the basis of the Feynman-Stueckelberg interpretation of anti-particles [3]), it was necessary to replace x^0 with a new order parameter. The Poincaré invariant parameter τ is formally similar to the Galilean invariant time in Newtonian theory, and its introduction permits the adaptation of many techniques from non-relativistic classical and quantum mechanics. In the symplectic mechanics which follows from this formulation, one writes classical Hamilton equations in the form

$$\frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} \quad \frac{dp^\mu}{d\tau} = -\frac{\partial K}{\partial x_\mu} \quad (1.1)$$

where

$$g_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \quad \text{and} \quad \mu, \nu = 0, \dots, 3, \quad (1.2)$$

and K is the analog of the Hamiltonian which generates system evolution according to τ . For the free particle, one may choose the Hamiltonian

$$K = \frac{p^\mu p_\mu}{2M}, \quad (1.3)$$

find the equations of motion

$$\frac{dt}{d\tau} = \frac{E}{M} \quad \frac{d\vec{x}}{d\tau} = \frac{\vec{p}}{M} \quad p^\mu = \text{constant}, \quad (1.4)$$

and recover $d\vec{x}/dt = \vec{p}/E$ in the usual form. Since m^2 is constant for the free particle, the proper time of the motion

$$ds^2 = d\vec{x}^2 - dt^2 = \frac{p^2}{M^2} d\tau^2 = -\frac{m^2}{M^2} d\tau^2 \quad (1.5)$$

is proportional to the invariant time τ in this case.

As in the non-relativistic case, one makes the transition from classical to quantum mechanics by regarding the Hamiltonian as the Hermitian generator of unitary τ evolution and writing

$$i \frac{\partial}{\partial \tau} \psi_\tau(x) = K \psi_\tau(x) \quad (1.6)$$

as a covariant Schrödinger equation with first order τ evolution. The squared magnitude of the wavefunction, $|\psi_\tau(x)|^2$, may be interpreted as a probability density, at τ , of finding the event at x . For the Hamiltonian (1.3), this density satisfies the conservation law

$$\partial_\tau |\psi_\tau(x)|^2 = \partial_\mu \left\{ \frac{i}{2M} [\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*] \right\} , \quad (1.7)$$

and so the integral of $|\psi_\tau(x)|^2$ over spacetime is conserved in τ . Eq. (1.7) corresponds to the current conservation law

$$\partial_\mu j^\mu + \frac{\partial \rho}{\partial \tau} = 0 , \quad (1.8)$$

where

$$j^\mu = -\frac{i}{2M} [\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*] \quad \text{and} \quad \rho = |\psi_\tau(x)|^2 . \quad (1.9)$$

The Green's function for the Schrödinger equation propagates the wavefunction monotonically from τ_1 to τ_2 , for $\tau_2 > \tau_1$ — although $x^0(\tau_2)$ need not be greater than $x^0(\tau_1)$.

In order to exploit the advantages of a covariant canonical formalism with invariant evolution, and the familiar methods of non-relativistic mechanics, both Schwinger and Feynman introduced an invariant time parameter — as a formal technique — in their work on quantum electrodynamics. In 1951, Schwinger [4] represented the Green's functions of the Dirac field as a parametric integral and formally transformed the Dirac problem into a dynamical theory in which the integration parameter acts as a proper time according to which a Hamiltonian operator generates the evolution of the system through spacetime. This method was the basis for his calculation of the vacuum polarization in an external electromagnetic field. Applying Schwinger's method to the Klein-Gordon equation, one obtains an equation for the Green's function (we take $\hbar = 1$ in the following)

$$G = \frac{1}{(p - eA)^2 + m^2 - i\epsilon} \quad (1.10)$$

given by

$$G(x, x') = \langle x | G | x' \rangle = i \int_0^\infty ds e^{-i(m^2 - i\epsilon)s} \langle x | e^{-i(p - eA)^2 s} | x' \rangle . \quad (1.11)$$

The function

$$G(x, x'; s) = \langle x(s) | x'(0) \rangle = \langle x | e^{-i(p - eA)^2 s} | x' \rangle \quad (1.12)$$

satisfies

$$i \frac{\partial}{\partial s} \langle x(s) | x'(0) \rangle = (p - eA)^2 \langle x(s) | x'(0) \rangle \quad (1.13)$$

with the boundary condition

$$\lim_{s \rightarrow 0} \langle x(s) | x'(0) \rangle = \delta^4(x - x') . \quad (1.14)$$

Schwinger regarded $x^\mu(s)$ and $\pi^\mu(s) = p^\mu(s) - eA^\mu(s)$ as operators, in a Heisenberg picture, which satisfy

$$\begin{aligned} [x^\mu, \pi^\nu] &= ig^{\mu\nu} & [\pi^\mu, \pi^\nu] &= ieF^{\mu\nu} \\ i[x^\mu, K] &= -\frac{\partial x^\mu}{\partial s} & i[\pi^\mu, K] &= -\frac{\partial \pi^\mu}{\partial s} . \end{aligned} \quad (1.15)$$

Equivalently, DeWitt [5] regarded (1.13) as defining the Green's function for the Schrödinger equation

$$i \frac{\partial}{\partial s} \psi_s(x) = K \psi_s(x) = (p - eA)^2 \psi_s(x) , \quad (1.16)$$

which (with the inclusion of a local metric tensor) he used for quantum mechanical calculations in curved space. The mini superspace formulation under investigation recently are of this type. Notice that (1.16) is in the form written by Stueckelberg in (1.6), using the Hamiltonian of (1.3) with $2M = 1$ and the minimal substitution $p^\mu \rightarrow p^\mu - eA^\mu$.

Feynman [6] used an expression identical to (1.16) in his derivation of the path integral for the Klein-Gordon equation. He regarded the integration of the Green's function with the weight $e^{-im^2 s}$, as the requirement (see also Nambu [7]) that asymptotic solutions of the Schrödinger equation be stationary eigenstates of the mass operator $i\partial_\tau$. From this point of view, one picks the mass eigenvalue by extending the lower limit of integration in (1.11) from 0 to $-\infty$, and adding the requirement that $G(x, x'; s) = 0$ for $s < 0$. It is worth noting [6] that the usual Feynman propagator $\Delta_F(x - x')$ emerges naturally from the classical causality condition of retarded propagation (in which s is the order parameter).

To Schwinger, the principal advantage of the proper time method is that the physical interactions are independent of the evolution parameter and so the symmetries of the system are preserved. Therefore [4], the method provides a natural approach to perturbation theory and regularization. In deriving the vacuum polarization, Schwinger found that all quantities remained finite as long as the lower limit of integration in (1.11) is taken to be $s_0 > 0$. Ball [8] demonstrates that the proper time method includes the usual regularization techniques as subcases and provides the specific correspondence with Pauli-Villars, point-splitting, zeta function, and dimensional regularization. Lüscher [9] used the proper time method to obtain the dimensional regularization in the presence of large background fields. The Casimir

effect [10] and the harmonic oscillator [11] partition function have recently been studied by inserting the factor s^ν into (1.11) for ν large enough to regularize the integral.

Covariant Two-Body Mechanics With Interactions

In 1973, Horwitz and Piron [12] constructed a canonical formalism for the relativistic classical and quantum mechanics of many particles. In order to formulate a generalized Hamilton's principle, they introduce a Poincaré invariant evolution parameter τ , which we shall call the *world time* and regard as corresponding to the ordering relation of successive events in spacetime. This τ is therefore similar to the parameter in the formalisms of Stueckelberg, Schwinger, and Feynman, except that it is regarded as a true physical time, with the status of the Newtonian time in non-relativistic mechanics. Moreover, by introducing two-body potentials between particles, Horwitz and Piron implicitly relaxed the requirement of a constant proportionality between the *world time* and the proper time of the classical particle motions. Thus, particles will not remain on mass-shell.

The two-body Hamiltonian in the Horwitz-Piron theory is

$$K = \frac{p_{1\mu}p_1^\mu}{2M_1} + \frac{p_{2\mu}p_2^\mu}{2M_2} + V(x_1, x_2) \quad (1.17)$$

and in the case of non-trivial interaction, the p^μ will not be constant and the particle masses become dynamical quantities. Non-relativistic central force problems may be generalized to covariant form by taking

$$V(x_1, x_2) = V(\rho) \quad \text{where} \quad \rho = \sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^2 - (t_1 - t_2)^2} . \quad (1.18)$$

Since $t_1 \rightarrow t_2$ in the Galilean limit ($\frac{dt_i}{d\tau} = \frac{E_i}{m_i c^2} \sim 1$), $V(\rho) \rightarrow V(r)$ for the corresponding non-relativistic problem. As in the non-relativistic problem, the two-body Hamiltonian is quadratic in the momenta, and one may separate variables of the center of mass motion and relative motion,

$$K = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m} + V(\rho) \equiv \frac{P^\mu P_\mu}{2M} + K_{rel}, \quad (1.19)$$

where

$$\begin{aligned} P^\mu &= p_1^\mu + p_2^\mu & M &= M_1 + M_2 \\ p^\mu &= (M_2 p_1^\mu - M_1 p_2^\mu)/M & m &= M_1 M_2 / M. \end{aligned} \quad (1.20)$$

Solutions of the Schrödinger equation have been found for the relativistic bound state [13, 14] and for scattering potentials [15]. For bound states, the mass spectrum coincides with the

non-relativistic Schrödinger energy spectrum, and so it follows that for small excitations, the corresponding energy spectrum is that of the non-relativistic Schrödinger theory with relativistic corrections. To obtain these spectra, one must choose a spacelike unit vector n_μ and restrict the support of the eigenfunctions in spacetime to the subspace of the Minkowski measure space for which the component of the relative coordinate $x_1^\mu - x_2^\mu$ normal to n_μ is spacelike. The restricted space is transitive and invariant under the $O(2,1)$ subgroup of $O(3,1)$ leaving n_μ invariant and translations along n_μ . Mathematically, this restriction is related to the existence of discrete unitary representations of the Lorentz group in this subspace [16]. Physically, this restriction leads to a lowering of the mass spectrum (compare [17]). It was shown in [13] that the eigenfunctions of K_{rel} form irreducible representations of $SU(1,1)$ — in the double covering of $O(2,1)$ — parameterized by the spacelike vector n_μ stabilized by this particular $O(2,1)$. In [14], an induced representation of $SL(2,C)$ was constructed, by applying the Lorentz group to the coordinates x^μ of the restricted space and the frame orientation n_μ , and studying the action on these wavefunctions.

In [18], the selection rules for dipole radiation from these states are calculated and shown to be identical with those of the usual non-relativistic theory, but with manifestly covariant interpretation. Moreover, it is shown that the change in the magnetic quantum number corresponds to a change in the orientation of n_μ with respect to the polarization of the emitted or absorbed photon. The group theoretical aspects of this bound state recoil are discussed in the context of the induced representation of $SL(2,C)$.

In [19], we provide a derivation of the normal Zeeman effect for the bound state, which requires n_μ to become a dynamical quantity. We begin with a discussion of the classical $O(3,1)$ in the induced representation and obtain the group generators, which coincide with those of [14], when the momenta are understood as derivatives in the Poisson bracket sense. We construct a classical Lagrangian, in which n_μ plays an explicit dynamical role, and show that the generators are conserved. We then construct the Hamiltonian, which may be unambiguously quantized and made locally gauge invariant. Finally, it is shown that an external gauge field representing a constant magnetic field induces, as a first order perturbation, a mass level splitting corresponding to the usual non-relativistic expression.

Local Gauge Theory

In the context of the covariant mechanics of Horwitz and Piron, Saad, Horwitz, and Arshan-

sky have argued [20] that the local gauge function of the field should include dependence on τ , as well as on the spacetime coordinates. This requirement of full gauge covariance leads to a theory of five gauge compensation fields, which differs in significant aspects from conventional electrodynamics, but whose zero modes coincide with the Maxwell theory (see also [21]).

Under local gauge transformations of the form

$$\psi(x, \tau) \rightarrow e^{ie_0\Lambda(x, \tau)}\psi(x, \tau) \quad (1.21)$$

the equation

$$(i\partial_\tau + e_0a_5)\psi(x, \tau) = \frac{1}{2M}(p^\mu - e_0a^\mu)(p_\mu - e_0a_\mu)\psi(x, \tau) \quad (1.22)$$

is invariant, when the compensation fields transform as

$$a_\mu(x, \tau) \rightarrow a_\mu(x, \tau) + \partial_\mu\Lambda(x, \tau) \quad a_5(x, \tau) \rightarrow a_5(x, \tau) + \partial_\tau\Lambda(x, \tau) , \quad (1.23)$$

where the potentials $a_\mu(x, \tau)$ and $a_5(x, \tau)$ must now also depend explicitly on the *world time* τ . This Schrödinger equation (1.22) leads (as for (1.8)) to the five dimensional conserved current

$$\partial_\mu j^\mu + \partial_\tau j^5 = 0 \quad (1.24)$$

where

$$j^5 \equiv \rho = |\psi(x, \tau)|^2 \quad j^\mu = \frac{-i}{2M} \left\{ \psi^* (\partial^\mu - ie_0a^\mu) \psi - \psi (\partial^\mu + ie_0a^\mu) \psi^* \right\} , \quad (1.25)$$

so that, as in Stueckelberg's formulation, $|\psi(x, \tau)|^2$ may be interpreted as a probability density. The current conservation law may be written as $\partial_\alpha j^\alpha = 0$, where the index convention is now

$$\lambda, \mu, \nu = 0, 1, 2, 3 \quad \text{and} \quad \alpha, \beta, \gamma = 0, 1, 2, 3, 5 \quad (1.26)$$

and the parameter τ is formally designated x^5 , so that $\partial_\tau = \partial_5$.

In [22], we obtain the classical Lorentz force, by using (1.22) to write the classical Hamiltonian and Lagrangian. The result,

$$M \ddot{x}^\mu = e_0 f^\mu{}_\alpha(x, \tau) \dot{x}^\alpha \quad \frac{d}{d\tau} \left(-\frac{1}{2} M \dot{x}^2 \right) = e_0 f_{5\alpha} \dot{x}^\alpha , \quad (1.27)$$

where $f_{\alpha\beta}$ is given by the gauge invariant quantity

$$f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha , \quad (1.28)$$

and where we use the fact that $\dot{x}^5 \equiv 1$, shows that the new field strength tensor components $f_{5\alpha}$ act on the classical particle motions in this theory. It follows from the second of (1.27) that mass need not be conserved in this theory, even at the classical level, and that pair annihilation is classically permitted [1]. It was shown in reference [22], by a study of the energy-momentum-mass tensor for the classical motions, that the total energy, momentum, and mass of the particles plus fields are conserved.

The Schrödinger equation (1.22) may be derived by variation of the action

$$S = \int d^4x d\tau \left\{ \psi^* (i\partial_\tau + e_0 a_5) \psi - \frac{1}{2M} \psi^* (p_\mu - e_0 a_\mu) (p^\mu - e_0 a^\mu) \psi - \frac{\lambda}{4} f_{\alpha\beta} f^{\alpha\beta} \right\} \quad (1.29)$$

to which Sa'ad, et. al. have added a kinetic term for the fields, formed from the gauge invariant quantity $f_{\alpha\beta}$. In writing the kinetic term, one must formally raise the index $\beta = 5$ in the term

$$f_{\mu 5} = \partial_\mu a_5 - \partial_\tau a_\mu, \quad (1.30)$$

and Sa'ad, et. al. argue that this term suggests a higher symmetry. Since this symmetry must contain $O(3,1)$, it can be $O(4,1)$ or $O(3,2)$, which correspond respectively to a signature $g^{55} = \sigma = \pm 1$, and they write the metric for the free field as

$$g^{\alpha\beta} = \text{diag}(-1, 1, 1, 1, \sigma). \quad (1.31)$$

Varying the action (1.29) with respect to the gauge fields, the equations of motion are found to be

$$\partial_\beta f^{\alpha\beta} = \frac{e_0}{\lambda} j^\alpha = e j^\alpha \quad (1.32)$$

$$\epsilon^{\alpha\beta\gamma\delta\epsilon} \partial_\alpha f_{\beta\gamma} = 0 \quad (1.33)$$

where j^α is given by (1.25). As we show below, λ and e_0 are dimensional constants; one identifies e_0/λ as the dimensionless Maxwell charge e . The sourceless gauge field equations inherit the formal five dimensional symmetry of the free gauge field Lagrangian, while the *physical* Lorentz covariance of the matter currents breaks the $O(4,1)$ or $O(3,2)$ symmetry of the free fields to $O(3,1)$. Nevertheless, the wave equation associated with the fields is [20]

$$\partial_\alpha \partial^\alpha f^{\beta\gamma} = (\partial_\mu \partial^\mu + \partial_\tau \partial^\tau) f^{\beta\gamma} = (\partial_\mu \partial^\mu + \sigma \partial_\tau^2) f^{\beta\gamma} = -e(\partial^\alpha j^\beta - \partial^\beta j^\alpha), \quad (1.34)$$

and the causal properties of the (free) Green's functions for the operator on the left hand side of (1.34) reflect the higher symmetry. The dependence of the wave equation on the signature

σ of ∂_τ implies that these causal properties will be different for the symmetry groups $O(3,2)$ and $O(4,1)$.

In [23], we derive the Green's functions; the $O(3,2)$ case supports spacelike and lightlike correlations through spacetime, without permitting superluminal transmission of information. The $O(4,1)$ case contains timelike and lightlike correlations; the timelike correlations similarly do not carry information in the usual sense.

Under the boundary conditions $j^5 \rightarrow 0$, pointwise, as $\tau \rightarrow \pm\infty$, integration of (1.25) over τ , leads to $\partial_\mu J^\mu = 0$, where

$$J^\mu(x) = \int_{-\infty}^{\infty} d\tau j^\mu(x, \tau) \quad (1.35)$$

so that we may identify J^μ as the source of the Maxwell field. Similarly, under the boundary conditions $f^{5\mu} \rightarrow 0$ pointwise in x as $\tau \rightarrow \pm\infty$, integration of (1.32) over τ leads to the Maxwell equations, in the form,

$$\partial_\nu F^{\mu\nu} = eJ^\mu \quad \epsilon^{\mu\nu\rho\lambda} \partial_\mu F_{\nu\rho} = 0 \quad (1.36)$$

where

$$F^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau f^{\mu\nu}(x, \tau) \quad \text{and} \quad A^\mu(x) = \int_{-\infty}^{\infty} d\tau a^\mu(x, \tau) \quad (1.37)$$

so that $a^\alpha(x, \tau)$ has been called the pre-Maxwell field. It follows from (1.37) that e_0 and λ have dimensions of length.

In the pre-Maxwell theory, interactions take place between events in spacetime rather than between worldlines. The resulting system of equations is integrable. Each event, occurring at τ , induces a current density in spacetime which, for free particles (and hence asymptotically), disperses for large τ , and the continuity equation (1.24) states that these current densities evolve as the event density j^5 progresses through spacetime as a function of τ . As noted above, if $j^5 \rightarrow 0$ as $|\tau| \rightarrow \infty$, then j^μ may be identified with the Maxwell current when integrated over τ . This integration has been called concatenation [24] and provides the link between the event along a worldline and the notion of a particle, whose support is the entire worldline. Concatenation is evidently related to the integration performed in the Schwinger proper time method, and following Feynman's interpretation, imposes the requirement that the Maxwell electromagnetic field be the zero mode with respect to the conjugate mass variable. The Maxwell theory thus has the character of an *equilibrium limit*

of the microscopic pre-Maxwell theory. Shnerb and Horwitz [25] have given an interpretation of the dimensional constant λ as a coherence length. In this sense, the Maxwell theory is a correlation limit of the pre-Maxwell theory, in which it is properly contained. Frastai and Horwitz [26] have shown that close to this limit, the strongest singularities of one-loop diagrams are regularized in a manner similar to that of Pauli and Villars [27].

Feynman's Approach to the Foundations of Gauge Theory

In 1948, Feynman showed Dyson how the Lorentz force law and homogeneous Maxwell equations could be derived from commutation relations among Euclidean coordinates and velocities, without reference to an action or variational principle. When Dyson published the work in 1990 [28], several authors [29, 30, 31, 32] noted that the derived equations have only Galilean symmetry and so are not actually the Maxwell theory. In a more recent paper, Tanimura [33] generalized Feynman's derivation to a Lorentz covariant form with scalar evolution parameter, and obtained an expression for the Lorentz force which appears to be consistent with relativistic kinematics and relates the force to the Maxwell field in the usual manner. However, Tanimura's derivation does not lead to the usual Maxwell theory either, but rather to the off-shell pre-Maxwell theory described here [34]. Hojman and Shepley [31] proved that the existence of commutation relations is a strong assumption, sufficient to determine the corresponding action, which for Feynman's derivation is of Newtonian form. In [34], we examine Tanimura's derivation in the framework of the proper time method, and use the technique of Hojman and Shepley to study the unconstrained commutation relations. In this context, we explain Tanimura's observations that the invariant evolution parameter cannot be consistently identified with the proper time of the particle motion, and that the derivation cannot be made reparameterization invariant. Using the techniques of Tanimura and of Hojman and Shepley, we obtain the form of the pre-Maxwell theory in a background curved space and in the presence of a classical non-Abelian gauge field.

Off-Shell Quantum Electrodynamics

Shnerb and Horwitz [25] have presented a consistent canonical quantization of off-shell electromagnetism which preserves the theory's formal five dimensional symmetry, using a bosonic gauge fixing method. Frastai and Horwitz [26] used a path integral quantization to study the theory in the near on-shell limit. In this paper, we develop the methods required to apply off-shell quantum electrodynamics as a scattering theory, and calculate certain ele-

mentary processes. In Section 2, we perform a canonical quantization of the interacting off-shell theory, which takes advantage the fact that it is a parameterized evolution theory. We use a procedure advocated by Jackiw [35] for theories in which the action can be made linear in time derivatives, and obtain results in complete agreement with those of Shnerb and Horwitz. For conventional quantum theories, in which the momentum is related to a t -derivative, putting the action into first order form may break manifest Lorentz covariance. For the off-shell theory, in which the classical velocity is a τ -derivative, Jackiw's approach is natural and manifestly covariant throughout. In Section 3, we show how Jackiw's approach leads to a path integral quantization in a equally natural way. Having identified the essential degrees of freedom, we perform the detailed Fourier expansion of the operators of the free fields, required in perturbation theory. In Section 4, we carry out a canonical quantization of the free off-shell matter field, and provide an expansion of the field operators in terms of annihilation operators in momentum space. We use these expansions to show that the vacuum expectation value of τ -ordered fields is precisely the Stueckelberg-Schrödinger equation Green's function, with Schwinger-Feynman boundary conditions. In Section 5, we carry out the canonical quantization of the five dimensional electromagnetic field and the expansion of field operators in momentum space. This field has three independent polarizations, one of which decouples in the Maxwell (zero mode) limit. We show that vacuum expectation value of products of two τ -ordered fields is the Feynman Green's function for the wave equation, and that it preserves the gauge condition. In Section 6, we develop the perturbation theory for the S-matrix in the interaction picture. In order to provide the link with the usual formulas for the perturbation expansion of Green's functions, we adapt the LSZ reduction formulas to the off-shell theory. We then derive the Feynman rules. In Section 7, we derive the scattering cross-section in terms of the transition amplitude, and demonstrate the relationship of this expression to the usual cross-section. In Section 8, we use the machinery of the previous sections to calculate the amplitudes for Compton and Møller scattering. We analyze the Møller scattering amplitude in detail and examine the qualitative deviation from the corresponding amplitude calculated in conventional QED. In Section 9, we consider the problem of renormalization of the off-theory. We derive the Ward identity associated with current conservation and show that it connects the matter field propagators with both the 3-particle and the 4-particle vertex functions. We conclude that the off-shell theory is renormalizable, when a cut-off is placed on the mass of the off-shell photons. This cut-off nevertheless preserves the invariances of the original theory.

2 Canonical Quantization of the Interacting Theory

Following [20] we take the action for off-shell electromagnetism to be

$$S = \int d^4x d\tau \left\{ \psi^* (i\partial_\tau + e_0 a_5) \psi - \frac{1}{2M} \psi^* (-i\partial_\mu - e_0 a_\mu) (-i\partial^\mu - e_0 a^\mu) \psi - \frac{\lambda}{4} f_{\alpha\beta} f^{\alpha\beta} \right\} \quad (2.1)$$

where

$$\mu, \nu = 0, \dots, 3 \quad \text{and} \quad \alpha, \beta = 0, \dots, 3, 5 \quad (2.2)$$

and

$$a_5 = \sigma a^5 \quad g^{\alpha\beta} = \text{diag}(-1, 1, 1, 1, \sigma). \quad (2.3)$$

We now show that the action in (2.1) is hermitian, up to surface terms which may be discarded from the action. We regard 4-divergences as vanishing under spacetime integration because of their support properties, while τ -derivatives vanish *independently* under τ -integration because of Riemann-Lebesgue oscillations (the asymptotic mass dependence of the fields $\exp\{-i(m^2/2M)\tau\}$ varies arbitrarily rapidly for large τ).

Thus, for the τ -derivative terms in (2.1),

$$\begin{aligned} \frac{1}{2} [\psi^* (i\partial_\tau + e_0 a_5) \psi + \text{h.c.}] &= \frac{i}{2} [\psi^* \dot{\psi} - \dot{\psi}^* \psi] + e_0 a_5 \psi^* \psi \\ &= i\psi^* \dot{\psi} + e_0 a_5 \psi^* \psi + \partial_\tau (\psi^* \psi) \end{aligned} \quad (2.4)$$

where $\dot{\psi} = \partial\psi/\partial\tau$. We will henceforth regard (2.1) as hermitian.

In Dirac's method [36] of quantization for gauge theories, we would form the Hamiltonian from (2.1), including a momentum π_5 conjugate to a^5 and a Lagrange multiplier to enforce the primary constraint $\pi_5 = 0$ (in [25], this is accomplished through the gauge fixing term). The secondary constraint (that the primary constraint commute with the Hamiltonian) would lead to the Gauss Law for the off-shell theory, which is the $\alpha = 5$ term of (1.32).

We will follow a quantization scheme advocated by Jackiw [35], in which one first eliminates the constraint from the Lagrangian, and then constructs the Hamiltonian from the unconstrained degrees of freedom. In order to perform the transformation which eliminates the constrained degrees of freedom, it is necessary that the action be made linear in the time derivatives. For conventional quantum theories, in which the field's t -derivative carries the

0th Lorentz index, the first order form of the action will not be manifestly covariant, and transformation properties must be checked after quantization (see also [37]). In off-shell electromagnetism, this method singles out the τ -derivatives of the fields, so that the method is $O(3,1)$ -covariant. The loss of the formal five-dimensional symmetry of the free off-shell electromagnetic field does not constitute a loss of generality for the interacting theory, since this higher symmetry is not a property of the matter field. Lorentz covariance is maintained throughout, in a manner consistent with the original spirit of the proper time method.

In Jackiw's method, a choice of gauge is made implicitly by solving the constraints and introducing a decomposition of the fields consistent with that solution. The elimination of the longitudinal polarizations is carried out as an application of the Darboux theorem, diagonalizing the Hamiltonian. This procedure is evidently related to the quantization method proposed by Fermi in 1932 [38]. Applying Fermi's method to (2.1), we would perform a gauge transformation which guarantees the condition $\partial_\mu a^\mu = 0$, and leads to the Gauss law for the component a_5 . The presence of sources in the action would require the decomposition of the fields into longitudinal terms induced by the sources and transverse propagating terms with vanishing four-divergence. We could then argue that the longitudinal fields do not satisfy wave equations and need not be quantized; this would leave the unconstrained transverse fields in the action, and we would obtain a consistent canonical quantization. Elements of such a procedure may be recognized in the method presented below.

We now put the action into the required first order form. The kinetic term for the matter field is linear in ∂_τ by construction. In order to put the kinetic term for the gauge field into explicitly canonical form, we rewrite $f^{5\mu}f_{5\mu}$ as $f^{5\mu}(\partial_\tau a_\mu - \partial_\mu a_5)$ and take the quantity $f^{5\mu}$, to be independent of the fields a_α (this is a variant of the first order Lagrangian form for the usual electromagnetic field). Expanding the electromagnetic term,

$$\begin{aligned} f_{\alpha\beta}f^{\alpha\beta} &= f_{\mu\nu}f^{\mu\nu} + 2f^{5\mu}f_{5\mu} \\ &= f_{\mu\nu}f^{\mu\nu} + 2\sigma f^{5\mu}f^5{}_\mu \\ &= f_{\mu\nu}f^{\mu\nu} + 2\sigma \left[2(\sigma\partial_\tau a^\mu - \partial^\mu a^5)f^5{}_\mu - f^{5\mu}f^5{}_\mu \right] , \end{aligned} \quad (2.5)$$

when we vary the action with respect to $f^{5\mu}$, we recover its relationship to the a^α . Now, performing the integration by parts, we obtain

$$(\sigma\partial_\tau a^\mu - \partial^\mu a^5)f^5{}_\mu = \sigma(\partial_\tau a^\mu)f^5{}_\mu + a^5\partial^\mu f^5{}_\mu - \text{divergence} ; \quad (2.6)$$

we drop the divergence and introduce the notation

$$\epsilon^\mu = f^{5\mu} . \quad (2.7)$$

Using (2.6) and (2.7), the action becomes

$$\begin{aligned} S &= \int d^4x d\tau \left[\psi^* (i\partial_\tau + e_0 a_5) \psi - \frac{1}{2M} \psi^* (-i\partial_\mu - e_0 a_\mu) (-i\partial^\mu - e_0 a^\mu) \psi \right. \\ &\quad \left. - \frac{\lambda}{4} f_{\mu\nu} f^{\mu\nu} + \frac{\lambda\sigma}{2} \epsilon^\mu \epsilon_\mu - \lambda \epsilon_\mu \partial_\tau a^\mu - \lambda a_5 \partial^\mu \epsilon_\mu \right] \\ &= \int d^4x d\tau \left[i\psi^* \dot{\psi} - \lambda \epsilon_\mu \dot{a}^\mu - \frac{1}{2M} \psi^* (-i\partial_\mu - e_0 a_\mu) (-i\partial^\mu - e_0 a^\mu) \psi \right. \\ &\quad \left. - \frac{\lambda}{4} f_{\mu\nu} f^{\mu\nu} + \frac{\lambda\sigma}{2} \epsilon^\mu \epsilon_\mu + a_5 (e_0 \psi^* \psi - \lambda \partial^\mu \epsilon_\mu) \right] \end{aligned} \quad (2.8)$$

where in the second line we have collected terms in a_5 . Following Jackiw, we regard (2.8) as an action for the conjugate pairs $\{\psi^*, \psi\}$ and $\{\epsilon_\mu, a^\mu\}$, with a_5 playing the role of a Lagrange multiplier for the constraint*

$$e_0 \psi^* \psi - \lambda \partial^\mu \epsilon_\mu = 0 \quad \Rightarrow \quad \partial^\mu \epsilon_\mu = \frac{e_0}{\lambda} \psi^* \psi = e\rho, \quad (2.9)$$

which is just the Gauss law for the off-shell theory. The constraint equation (2.9) can be solved through the decomposition

$$\epsilon^\mu = (\epsilon_\perp)^\mu + e\partial^\mu [G\rho] \quad (2.10)$$

where

$$\partial_\mu (\epsilon_\perp)^\mu = 0 \quad (2.11)$$

and where $G\rho$ is a shorthand for the functional

$$[G\rho](x, \tau) = \int d^4y G(x - y) \rho(y, \tau) \quad (2.12)$$

in which we specify the Green's function

$$G(x - y) = \delta((x - y)^2) \quad \Rightarrow \quad \square G = 1. \quad (2.13)$$

Performing a similar decomposition of a^μ ,

$$a^\mu = (a_\perp)^\mu + \partial^\mu [G\Lambda] \quad \partial_\mu (a_\perp)^\mu = 0 \quad (2.14)$$

*It will be shown below that the resulting quantized fields satisfy equations corresponding to the classical theory in the $a_5 = 0$ gauge.

(by which we implicitly choose the gauge condition $\partial_\mu a^\mu = \Lambda$), the remaining terms of the theory are expressed as

$$\dot{a}^\mu = (\dot{a}_\perp)^\mu + \partial^\mu[G\dot{\Lambda}], \quad (2.15)$$

$$f^{\mu\nu} = \partial^\mu(a_\perp)^\nu - \partial^\nu(a_\perp)^\mu = (f_\perp)^{\mu\nu}, \quad (2.16)$$

$$-i\partial^\mu - e_0 a^\mu = -i\partial^\mu - e_0(a_\perp)^\mu - e_0\partial^\mu[G\dot{\Lambda}], \quad (2.17)$$

$$\begin{aligned} \epsilon^\mu \epsilon_\mu &= [(\epsilon_\perp)^\mu + e\partial^\mu[G\rho]][(\epsilon_\perp)_\mu + e\partial_\mu[G\rho]] \\ &= (\epsilon_\perp)^\mu(\epsilon_\perp)_\mu - e^2\rho[G\rho] + \text{divergence}, \end{aligned} \quad (2.18)$$

$$\epsilon_\mu \dot{a}^\mu = (\epsilon_\perp)_\mu(\dot{a}_\perp)^\mu - e\rho[G\dot{\Lambda}]. \quad (2.19)$$

In terms of this decomposition, the action becomes

$$\begin{aligned} S &= \int d^4x d\tau \left\{ i\psi^* \dot{\psi} - \lambda(\epsilon_\perp)_\mu(\dot{a}_\perp)^\mu + \lambda e\rho[G\dot{\Lambda}] \right. \\ &\quad - \frac{1}{2M} \psi^* (-i\partial_\mu - e_0(a_\perp)_\mu - e_0\partial_\mu[G\dot{\Lambda}])(-i\partial^\mu - e_0(a_\perp)^\mu - e_0\partial^\mu[G\dot{\Lambda}])\psi \\ &\quad \left. - \frac{\lambda}{4}(f_\perp)_{\mu\nu}(f_\perp)^{\mu\nu} + \frac{\lambda\sigma}{2}(\epsilon_\perp)^\mu(\epsilon_\perp)_\mu - \frac{\lambda\sigma}{2}e^2\rho[G\rho] \right\}. \end{aligned} \quad (2.20)$$

We now perform the gauge transformation

$$\psi \longrightarrow e^{ie_0[G\Lambda]}\psi \quad (2.21)$$

which entails

$$\dot{\psi} \longrightarrow e^{ie_0[G\Lambda]}[\dot{\psi} + ie_0[G\dot{\Lambda}]\psi] \quad i\psi^* \dot{\psi} \longrightarrow i\psi^* \dot{\psi} - e_0\rho[G\dot{\Lambda}] \quad (2.22)$$

and

$$\begin{aligned} (-i\partial^\mu - e_0(a_\perp)^\mu - e_0\partial^\mu[G\dot{\Lambda}])\psi &\longrightarrow (-i\partial^\mu - e_0(a_\perp)^\mu - e_0\partial^\mu[G\dot{\Lambda}])e^{ie_0[G\Lambda]}\psi \\ &= e^{ie_0[G\Lambda]}(-i\partial^\mu - e_0(a_\perp)^\mu - e_0\partial^\mu[G\dot{\Lambda}] + e_0\partial^\mu[G\dot{\Lambda}])\psi \\ &= e^{ie_0[G\Lambda]}(-i\partial^\mu - e_0(a_\perp)^\mu)\psi. \end{aligned} \quad (2.23)$$

This transforms the action to the form

$$\begin{aligned} S = \int d^4x d\tau \{ & i\psi^* \dot{\psi} - \lambda(\epsilon_\perp)_\mu (\dot{a}_\perp)^\mu - \frac{1}{2M} \psi^* (-i\partial_\mu - e_0(a_\perp)_\mu) (-i\partial^\mu - e_0(a_\perp)^\mu) \psi \\ & - \frac{\lambda}{4} (f_\perp)_{\mu\nu} (f_\perp)^{\mu\nu} + \frac{\lambda\sigma}{2} (\epsilon_\perp)^\mu (\epsilon_\perp)_\mu - \frac{\lambda\sigma}{2} e^2 \rho[G\rho] \} . \end{aligned} \quad (2.24)$$

Notice that (2.24) is an unconstrained functional (only the unconstrained field degrees of freedom are present) and so may be canonically quantized. Henceforth, we may drop the subscript \perp from the quantized gauge field variables and assume the transversality conditions (2.11) and (2.14) for the field operators.

The conjugate momenta are found from (2.24) to be

$$\pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^* \quad (2.25)$$

$$\pi_{a_\mu} = \frac{\partial \mathcal{L}}{\partial \dot{a}_\mu} = -\lambda \epsilon^\mu \quad (2.26)$$

from which we compute the Hamiltonian

$$\begin{aligned} \mathcal{K} &= \pi_\psi \dot{\psi} + \pi_{a_\mu} \dot{a}_\mu - \mathcal{L} \\ &= i\psi^* \dot{\psi} - \lambda \epsilon^\mu \dot{a}_\mu - \left\{ i\psi^* \dot{\psi} - \lambda \epsilon_\mu \dot{a}^\mu - \frac{1}{2M} [(i\partial_\mu - e_0 a_\mu) \psi^*] [(-i\partial^\mu - e_0 a^\mu) \psi] \right. \\ &\quad \left. - \frac{\lambda}{4} f_{\mu\nu} f^{\mu\nu} + \frac{\lambda\sigma}{2} \epsilon^\mu \epsilon_\mu - \frac{\lambda\sigma}{2} e^2 \rho[G\rho] \right\} . \\ &= \frac{1}{2M} [(i\partial_\mu - e_0 a_\mu) \psi^*] [(-i\partial^\mu - e_0 a^\mu) \psi] + \frac{\lambda}{4} f_{\mu\nu} f^{\mu\nu} - \frac{\lambda\sigma}{2} \epsilon^\mu \epsilon_\mu + \frac{\lambda\sigma}{2} e^2 \rho[G\rho]. \end{aligned} \quad (2.27)$$

We may now decompose (2.27) into the free Hamiltonian for the matter and the gauge fields and the interaction terms.

$$\mathcal{K} = \mathcal{K}_{\text{photon}} + \mathcal{K}_{\text{matter}} + \mathcal{K}_{\text{interaction}} \quad (2.28)$$

where

$$\mathcal{K}_{\text{photon}} = \frac{\lambda}{4} f_{\mu\nu} f^{\mu\nu} - \frac{\lambda\sigma}{2} \epsilon^\mu \epsilon_\mu \quad (2.29)$$

$$\mathcal{K}_{\text{matter}} = \frac{1}{2M} [\partial_\mu \psi^*] [\partial^\mu \psi] \quad (2.30)$$

$$\mathcal{K}_{\text{interaction}} = \frac{ie_0}{2M} a_\mu (\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*) + \frac{e_0^2}{2M} a_\mu a^\mu |\psi|^2 + \frac{\lambda\sigma}{2} e^2 \rho[G\rho] \quad (2.31)$$

The last term in (2.31) has the form of a c-number energy density which represents the mass-energy equivalent required to assemble the matter field (see also [25]). From equations (1.27) and (1.32), one sees that the coupling $e_0 e = \lambda e^2$, which appears in this mass-energy density, is characteristic of the classical Lorentz force due to a charge distribution. On the other hand, we demonstrate below that the remaining terms in (2.31), which lead to 3-particle and 4-particle interactions in the Feynman diagrams, are connected by the Ward identity for the conserved current (1.24).

3 Path Integral Quantization

The first order action used in the previous section is naturally suited to path integral quantization. We write the path integral as

$$\mathcal{Z} = \frac{1}{\mathcal{N}} \int \mathcal{D}\psi^* \mathcal{D}\psi \mathcal{D}a_\mu \mathcal{D}a_5 \mathcal{D}\epsilon_\mu e^{iS} \quad (3.1)$$

where we use (2.8) for the action S. The integration over a_5 places the constraint (2.9) into the measure in the form of $\delta(e_0\psi^*\psi - \lambda\partial^\mu\epsilon_\mu)$; we may, furthermore, insert the gauge fixing constraint $\delta(\partial_\mu a^\mu - \Lambda)$. By integrating over $\mathcal{D}\epsilon_\parallel$ and $\mathcal{D}a_\parallel$, the path integral becomes

$$\mathcal{Z} = \frac{1}{\mathcal{N}} \int \mathcal{D}\psi^* \mathcal{D}\psi \mathcal{D}(a_\perp)_\mu \mathcal{D}(\epsilon_\perp)_\mu e^{iS} \quad (3.2)$$

where S is now given by (2.20). Then, by carrying out the gauge transformation (2.21), which leaves the measure invariant because we have chosen some specific function Λ , we obtain the path integral

$$\mathcal{Z} = \frac{1}{\mathcal{N}} \int \mathcal{D}\psi^* \mathcal{D}\psi \mathcal{D}(a_\perp)_\mu \mathcal{D}(\epsilon_\perp)_\mu e^{iS} \quad (3.3)$$

in which S is the unconstrained action in (2.24). Since ϵ_\perp plays the role of a conjugate momentum, we may perform the Gaussian integration over $\mathcal{D}(\epsilon_\perp)_\mu$, which puts the path integral into the form

$$\mathcal{Z} = \frac{1}{\mathcal{N}} \int \mathcal{D}\psi^* \mathcal{D}\psi \mathcal{D}(a_\perp)_\mu e^{iS} \quad (3.4)$$

where the action S is given by

$$\begin{aligned} S = \int d^4x d\tau \left\{ i\psi^* \dot{\psi} - \frac{1}{2M} \psi^* (-i\partial_\mu - e_0(a_\perp)_\mu) (-i\partial^\mu - e_0(a_\perp)^\mu) \psi \right. \\ \left. - \frac{\lambda}{4} (f_\perp)_{\mu\nu} (f_\perp)^{\mu\nu} - \frac{\lambda\sigma}{2} (\dot{a}_\perp)_\mu (\dot{a}_\perp)^\mu - \frac{\lambda\sigma}{2} e^2 \rho [G\rho] \right\}. \end{aligned} \quad (3.5)$$

We expand

$$\begin{aligned}
-\frac{\lambda}{4}(f_\perp)_{\mu\nu}(f_\perp)^{\mu\nu} &= -\frac{\lambda}{4}(\partial_\mu a_\nu - \partial_\nu a_\mu)(\partial^\mu a^\nu - \partial^\nu a^\mu) \\
&= -\frac{\lambda}{2}[(\partial_\mu a_\nu)(\partial^\mu a^\nu) - (\partial_\mu a_\nu)(\partial^\nu a^\mu)] \\
&= \frac{\lambda}{2}a_\mu[g^{\mu\nu}\square - \partial^\mu\partial^\nu]a_\nu + \text{divergence}
\end{aligned} \tag{3.6}$$

and similarly,

$$(\dot{a}_\perp)_\mu(\dot{a}_\perp)^\mu = -(a_\perp)_\mu[g^{\mu\nu}\partial_\tau^2](a_\perp)_\nu + \text{divergence} \tag{3.7}$$

so that the action takes the form

$$\begin{aligned}
S &= \int d^4x d\tau \left\{ i\psi^*\dot{\psi} - \frac{1}{2M}\psi^*(-i\partial_\mu - e_0(a_\perp)_\mu)(-i\partial^\mu - e_0(a_\perp)^\mu)\psi \right. \\
&\quad \left. + \frac{\lambda}{2}(a_\perp)_\mu[g^{\mu\nu}\square + \sigma g^{\mu\nu}\partial_\tau^2 - \partial^\mu\partial^\nu](a_\perp)_\nu - \frac{\lambda\sigma}{2}e^2\rho[G\rho] \right\}.
\end{aligned} \tag{3.8}$$

In Section 5, we will show that the operator $[g^{\mu\nu}\square - \partial^\mu\partial^\nu]$ projects onto the transverse states.

Since only transverse states are present in (3.8), we may replace the action with

$$\begin{aligned}
S &= \int d^4x d\tau \left\{ i\psi^*\dot{\psi} - \frac{1}{2M}\psi^*(-i\partial_\mu - e_0(a_\perp)_\mu)(-i\partial^\mu - e_0(a_\perp)^\mu)\psi \right. \\
&\quad \left. + \frac{\lambda}{2}(a_\perp)_\mu[\square + \sigma\partial_\tau^2](a_\perp)^\mu - \frac{\lambda\sigma}{2}e^2\rho[G\rho] \right\},
\end{aligned} \tag{3.9}$$

and we recognize the “inverse propagator” $[\square + \sigma\partial_\tau^2]$ for the gauge fields. From this action, the Feynman rules for Green’s functions, which will be derived in Section 6 by canonical procedures, may be read-off directly.

4 Canonical Quantization of the Free Spinless Matter Field

We begin with the Hamiltonian density (2.30) and take the Hamiltonian operator to be

$$K = \int d^4x \mathcal{K}_{\text{matter}}. \tag{4.1}$$

The fields must satisfy the canonical equal- τ commutation relations,

$$[\psi(x, \tau), \pi_\psi(x', \tau)] = i\delta^4(x - x'), \tag{4.2}$$

which together with (2.25) leads to

$$[\psi(x, \tau), \psi^*(x', \tau)] = \delta^4(x - x'). \quad (4.3)$$

The field evolves dynamically according to the Heisenberg equation

$$i\partial_\tau \psi = [\psi, K], \quad (4.4)$$

and using equations (2.30), (4.1), (4.2), and (4.4), we find

$$\begin{aligned} i\partial_\tau \psi(x, \tau) &= \left[\psi(x, \tau), \frac{1}{2M} \int d^4x' \left(\partial'_\mu \psi^*(x', \tau) \right) (\partial'^\mu \psi(x', \tau)) \right] \\ &= -\frac{1}{2M} \int d^4x' [\psi(x, \tau), \psi^*(x', \tau) \partial'_\mu \partial'^\mu \psi(x', \tau)] \\ &= -\frac{1}{2M} \int d^4x' [\psi(x, \tau), \psi^*(x', \tau)] \partial'_\mu \partial'^\mu \psi(x', \tau) \\ &= -\frac{1}{2M} \int d^4x' \delta^4(x - x') \partial'_\mu \partial'^\mu \psi(x', \tau) \\ &= -\frac{1}{2M} \partial_\mu \partial^\mu \psi(x, \tau). \end{aligned} \quad (4.5)$$

Notice that (4.5) is the Schrödinger equation (1.22) for the field operator $\psi(x, \tau)$ in the absence of interaction, and because it is satisfied, we may perform the Fourier expansion

$$\psi(x, \tau) = \int \frac{d^4k}{(2\pi)^4} b(k) e^{i(k \cdot x - \kappa \tau)} \quad \psi^*(x, \tau) = \int \frac{d^4k}{(2\pi)^4} b^*(k) e^{-i(k \cdot x - \kappa \tau)} \quad (4.6)$$

where from (4.5) we see that $\kappa = k^2/2M$. From (4.2), we find that

$$[b(k), b^*(k')] = (2\pi)^4 \delta^4(k - k') \quad [b(k), b(k')] = [b^*(k), b^*(k')] = 0. \quad (4.7)$$

In terms of the Fourier expansion, the Hamiltonian is given by

$$\begin{aligned} K &= \int d^4x \frac{1}{2M} [\partial_\mu \psi^*] [\partial^\mu \psi] \\ &= \frac{1}{2M} \int d^4x \int \frac{d^4k}{(2\pi)^4} (-ik_\mu) b^*(k) e^{-i(k \cdot x - \kappa \tau)} \int \frac{d^4k'}{(2\pi)^4} (ik'_\mu) b(k') e^{i(k' \cdot x - \kappa' \tau)} \\ &= \frac{1}{2M} \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} b^*(k) b(k') (2\pi)^4 \delta^4(k - k') e^{i\kappa \tau} e^{-i\kappa' \tau} (k \cdot k') \\ &= \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{2M} b^*(k) b(k) \\ &= \int \frac{d^4k}{(2\pi)^4} \kappa b^*(k) b(k) \end{aligned} \quad (4.8)$$

Notice that the ground state mass for this Hamiltonian vanishes even without the need for normal ordering.

We write the Green's function $G_i(x, \tau)$ for the Schrödinger equation (4.5) in the integral form,

$$G(x, \tau) = \frac{1}{(2\pi)^5} \int_{C_i} d^4k d\kappa \frac{e^{i(k \cdot x - \kappa \tau)}}{\frac{1}{2M}k^2 - \kappa} \quad (4.9)$$

where the contour of κ integration C_i determines the boundary conditions. If C_i includes the interval $(-\infty, \infty)$, then $G_i(x, \tau)$ will satisfy the inhomogeneous equation

$$(i\partial_\tau + \frac{1}{2M}\partial^\mu\partial_\mu)G_i(x, \tau) = -\delta^4(x)\delta(\tau). \quad (4.10)$$

Recalling Schwinger's parametric representation of the Klein-Gordon Green's function and Feynman's observation that the choice of retarded contour in (4.9) (by displacing the pole into the lower half plane) is equivalent to choosing the Feynman contour for the τ -integrated propagator, we will take the Green's function for the matter field to be

$$G(x, \tau) = \frac{1}{(2\pi)^5} \int d^4k d\kappa \frac{e^{i(k \cdot x - \kappa \tau)}}{\frac{1}{2M}k^2 - \kappa - i\epsilon}. \quad (4.11)$$

First performing the κ -integration (the pole is at $\kappa = k^2/2M - i\epsilon$ in the lower half plane), we obtain zero for $\tau < 0$ (for which we must close in the upper half plane) and for $\tau > 0$ (we must close in the lower half plane), we obtain

$$\int d\kappa \frac{e^{-i\kappa\tau}}{\frac{1}{2M}k^2 - \kappa - i\epsilon} = 2\pi i \text{Res} \frac{e^{-i\kappa\tau}}{\frac{1}{2M}k^2 - \kappa - i\epsilon} \times \text{index of contour} = 2\pi i e^{-i(\frac{k^2}{2M} - i\epsilon)\tau} \theta(\tau). \quad (4.12)$$

Thus, we find that

$$G(x, \tau) = \frac{i}{(2\pi)^4} \int d^4k e^{i(k \cdot x - \frac{k^2}{2M}\tau + i\epsilon\tau)} \theta(\tau) \quad (4.13)$$

in which we see the retarded nature of $G(x, \tau)$ explicitly through $\theta(\tau)$. We easily verify that the τ -integral of the Green's function,

$$\begin{aligned} \int_{-\infty}^{\infty} d\tau e^{-i(m^2/2M)\tau} G(x, \tau) &= \frac{i}{(2\pi)^4} \int_0^{\infty} d\tau e^{-i(m^2/2M)\tau} \int d^4k e^{i(k \cdot x - \frac{k^2}{2M}\tau + i\epsilon\tau)} \\ &= i \int_0^{\infty} d\tau \int \frac{d^4k}{(2\pi)^4} e^{-i(m^2/2M + \frac{k^2}{2M} - i\epsilon)\tau} e^{ik \cdot x} \\ &= \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik \cdot x}}{\frac{1}{2M}(k^2 + m^2) - i\epsilon} \\ &= 2M \Delta_F(x) \end{aligned} \quad (4.14)$$

goes over to the Feynman propagator for a particle of mass m with an overall factor of $2M$.

On the other hand, we may consider the Green's function for the matter field as expressed through the vacuum expectation value of the (τ -ordered) operator products. Using the momentum expansions (4.6), we find that

$$\begin{aligned}
\langle 0 | T \psi(x_1, \tau_1) \psi^*(x_2, \tau_2) | 0 \rangle &= \\
&= \theta(\tau_1 - \tau_2) \langle 0 | \psi(x_1, \tau_1) \psi^*(x_2, \tau_2) | 0 \rangle + \theta(\tau_2 - \tau_1) \langle 0 | \psi^*(x_2, \tau_2) \psi(x_1, \tau_1) | 0 \rangle \\
&= \theta(\tau_1 - \tau_2) \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 k'}{(2\pi)^4} e^{-i(k \cdot x_2 - \frac{k^2}{2M} \tau_2)} e^{i(k' \cdot x_1 - \frac{k'^2}{2M} \tau_1)} \langle 0 | b(k') b^*(k) | 0 \rangle \\
&= \theta(\tau_1 - \tau_2) \int \frac{d^4 k}{(2\pi)^4} e^{i[k \cdot (x_1 - x_2) - \frac{k^2}{2M} (\tau_1 - \tau_2)]} \\
&= -iG(x_1 - x_2, \tau_1 - \tau_2)
\end{aligned} \tag{4.15}$$

where we have used $b(k)|0\rangle = 0$ and (4.7). We thus verify that the Green's function we have chosen in (4.11) is equal to the vacuum expectation value of the τ -ordered product of the operators, as required for the application of Wick's theorem in perturbation theory.

5 Quantization of the Free Gauge Field

We begin with the Hamiltonian obtained from (2.29) as

$$K = \int d^4 x \mathcal{K}_{\text{photon}}. \tag{5.1}$$

In order to evaluate the canonical commutation relations for the transverse fields, we first consider the projection operator

$$\delta_{\perp}^{\mu\nu}(x-y) = \int \frac{d^4 k}{(2\pi)^4} \left[g^{\mu\nu} - \frac{k^{\mu} k^{\nu}}{k^2} \right] e^{ik \cdot (x-y)} = \int \frac{d^4 k}{(2\pi)^4} \mathcal{P}^{\mu\nu}(k) e^{ik \cdot (x-y)}, \tag{5.2}$$

which projects onto the transverse part of a vector function. For a function $f_{\nu}(x, \tau)$ with Fourier transform $\tilde{f}_{\nu}(k, \tau)$,

$$\begin{aligned}
\int d^4 y \delta_{\perp}^{\mu\nu}(x-y) f_{\nu}(y, \tau) &= \int d^4 y \frac{d^4 k}{(2\pi)^4} d^4 k' \left[g^{\mu\nu} - \frac{k^{\mu} k^{\nu}}{k^2} \right] e^{ik \cdot (x-y)} \tilde{f}_{\nu}(k', \tau) e^{ik' \cdot y} \\
&= \int d^4 k \left[g^{\mu\nu} - \frac{k^{\mu} k^{\nu}}{k^2} \right] f_{\nu}(k, \tau) e^{ik \cdot x}
\end{aligned} \tag{5.3}$$

which clearly satisfies $\partial^{\mu} f_{\mu}(x, \tau) = 0$. So we may represent the transverse fields as

$$a_{\perp}^{\mu}(x, \tau) = [\delta_{\perp}^{\mu\rho} a_{\rho}](x, \tau) \quad \epsilon_{\perp}^{\mu}(x, \tau) = [\delta_{\perp}^{\mu\rho} \epsilon_{\rho}](x, \tau). \tag{5.4}$$

Canonical quantization requires the equal- τ commutation relations for the field components

$$[a^\mu(x, \tau), \pi_{a_\nu}(y, \tau)] = ig^{\mu\nu} \delta^4(x - y) \quad (5.5)$$

which, using (2.26) for π_{a_ν} , becomes

$$[a^\mu(x, \tau), \epsilon^\nu(y, \tau)] = -\frac{i}{\lambda} g^{\mu\nu} \delta^4(x - y). \quad (5.6)$$

So for the transverse field components, we find

$$\begin{aligned} [a_\perp^\mu(x, \tau), \epsilon_\perp^\nu(y, \tau)] &= [[\delta_\perp^{\mu\rho} a_\rho](x, \tau), [\delta_\perp^{\nu\sigma} \epsilon_\sigma](y, \tau)] \\ &= \frac{-i}{\lambda} g_{\rho\sigma} [\delta_\perp^{\mu\rho} \delta_\perp^{\nu\sigma} \delta^4](x - y) \\ &= \frac{-i}{\lambda} \delta_\perp^{\mu\nu} \delta^4(x - y) \end{aligned} \quad (5.7)$$

We may drop the subscript \perp from the field variables provided that we use (5.7) as the canonical commutation relation. This relation insures that the Gauss Law constraint (2.9) commutes with the other variables of the theory, and so, with the Hamiltonian (5.1).

The Heisenberg equations for the fields a^μ are

$$\begin{aligned} i\partial_\tau a^\mu(x, \tau) &= [a^\mu(x, \tau), K] \\ &= [a^\mu(x, \tau), \int d^4y (\frac{\lambda}{4} f_{\rho\nu} f^{\rho\nu} - \frac{\lambda\sigma}{2} \epsilon^\rho \epsilon_\rho)] \\ &= -\frac{\lambda\sigma}{2} \int d^4y \{ [a^\mu(x, \tau), \epsilon^\rho(y, \tau)] \epsilon_\rho(y, \tau) + \epsilon^\rho(y, \tau) [a^\mu(x, \tau), \epsilon_\rho(y, \tau)] \} \\ &= -\frac{\lambda\sigma}{2} [-2\frac{i}{\lambda} \epsilon^\mu(x, \tau)], \end{aligned}$$

and we find that

$$\partial_\tau a^\mu(x, \tau) = \sigma \epsilon^\mu(x, \tau) = \sigma f^{5\mu} = f_5{}^\mu \quad (5.8)$$

which agrees with the classical definition of $f^{5\mu}$ in the gauge $a_5 = 0$. The Heisenberg equation for the fields ϵ^μ are

$$\begin{aligned} i\partial_\tau \epsilon^\mu(x, \tau) &= [\epsilon^\mu(x, \tau), K] \\ &= [\epsilon^\mu(x, \tau), \int d^4y \frac{\lambda}{4} f_{\mu\nu} f^{\mu\nu}] \\ &= \frac{\lambda}{4} \int d^4y \{ [\epsilon^\mu(x, \tau), (\partial_\lambda a_\nu - \partial_\nu a_\lambda)] f^{\lambda\nu} + f^{\lambda\nu} [\epsilon^\mu(x, \tau), (\partial_\lambda a_\nu - \partial_\nu a_\lambda)] \} \\ &= \frac{\lambda}{4} \int d^4y \{ \partial_\lambda [\epsilon^\mu, a_\nu] f^{\lambda\nu} - \partial_\nu [\epsilon^\mu, a_\lambda] f^{\lambda\nu} + f^{\lambda\nu} \partial_\lambda [\epsilon^\mu, a_\nu] - f^{\lambda\nu} \partial_\nu [\epsilon^\mu, a_\lambda] \} \\ &= \frac{i}{4} \int d^4y \{ 2f^{\lambda\mu} \partial_\lambda \delta^4(x - y) - 2f^{\mu\lambda} \partial_\lambda \delta^4(x - y) \} \\ &= -i\partial_\lambda f^{\lambda\mu}(x, \tau). \end{aligned} \quad (5.9)$$

Since $\epsilon^\mu = f^{5\mu}$, (5.9) can be written

$$\partial_\lambda f^{\lambda\mu} + \partial_\tau f^{5\mu} = 0 \quad (5.10)$$

which may be combined with (2.11) to write

$$\partial_\alpha f^{\alpha\beta} = 0 \quad (5.11)$$

for $\alpha, \beta = 0, \dots, 3, 5$. These are half of the pre-Maxwell field equations; the other pre-Maxwell equations follow from the definition of $f^{\mu\nu}$ and (5.8). By virtue of (2.14), (5.10) may be written in the form

$$0 = \partial_\tau f^{5\mu} + \partial_\lambda (\partial^\lambda a^\mu - \partial^\mu a^\lambda) = \partial_\tau f^{5\mu} + \partial_\lambda \partial^\lambda a^\mu - \partial^\mu \partial_\lambda a^\lambda = \partial_\tau f^{5\mu} + \partial_\lambda \partial^\lambda a^\mu. \quad (5.12)$$

Combining (5.12) with (5.8), we obtain the wave equation,

$$0 = \partial_\lambda \partial^\lambda a^\mu + \partial_\tau \epsilon^\mu = \partial_\lambda \partial^\lambda a^\mu + \partial_\tau (\sigma \partial_\tau a^\mu) = (\partial_\lambda \partial^\lambda + \sigma \partial_\tau^2) a^\mu. \quad (5.13)$$

Therefore, we may perform a Fourier expansion of the field operator,

$$a^\mu(x, \tau) = \sum_{s=\text{polarizations}} \int \frac{d^4 k}{2\kappa} \left[\varepsilon_s^\mu a(k, s) e^{i(k \cdot x + \sigma \kappa \tau)} + \varepsilon_s^{\mu*} a^*(k, s) e^{-i(k \cdot x + \sigma \kappa \tau)} \right] \quad (5.14)$$

where the five-dimensional mass shell condition is

$$\kappa = \sqrt{-\sigma k^2}. \quad (5.15)$$

Evidently, for free fields, the condition $-\sigma k^2 > 0$ is required to prevent divergence at $\tau \rightarrow \pm\infty$, however this condition need not apply for virtual photons. Using (5.8), ϵ^μ is given by

$$\epsilon^\mu(x, \tau) = i \sum_{s=\text{polarizations}} \int \frac{d^4 k}{2\kappa} \kappa \left[\varepsilon_s^\mu a(k, s) e^{i(k \cdot x + \sigma \kappa \tau)} - \varepsilon_s^{\mu*} a^*(k, s) e^{-i(k \cdot x + \sigma \kappa \tau)} \right]. \quad (5.16)$$

We must choose polarization basis states for the field vectors. In momentum space (2.14) becomes $k_\mu a^\mu = 0$, so that there will be three independent polarizations (see also [39]). To choose the polarizations in a covariant way, we begin by choosing an arbitrary timelike vector n^μ ($n^2 = -1$). There are then two orthonormal spacelike vectors, ε_1 and ε_2 , which satisfy the conditions

$$n \cdot \varepsilon_1 = n \cdot \varepsilon_2 = k \cdot \varepsilon_1 = k \cdot \varepsilon_2 = 0, \quad (5.17)$$

which may be constructed in the following way. Since $n^2 = -1$,

$$(\vec{n})^2 - (n^0)^2 = -1 \implies n^0 = \pm \sqrt{(\vec{n})^2 + 1} \quad (5.18)$$

so that taking

$$\varepsilon_{1,2}^0 = \frac{\vec{n} \cdot \vec{\varepsilon}_{1,2}}{\sqrt{(\vec{n})^2 + 1}} \quad (5.19)$$

guarantees that $n \cdot \varepsilon_{1,2} = 0$. We now require that

$$0 = k \cdot \varepsilon_{1,2} = \vec{k} \cdot \vec{\varepsilon}_{1,2} - (k^0) \varepsilon_{1,2}^0 = \left[\vec{k} - \frac{k^0}{\sqrt{(\vec{n})^2 + 1}} \vec{n} \right] \cdot \vec{\varepsilon}_{1,2}. \quad (5.20)$$

Since two orthogonal 3-vectors can be found which satisfy (5.20), this establishes that (5.17) can also be satisfied. We normalize $\varepsilon_{1,2}$ so that

$$\varepsilon_1 \cdot \varepsilon_2 = 0 \quad (\varepsilon_1)^2 = (\varepsilon_2)^2 = 1 \quad (5.21)$$

To form the third polarization, which we will denote ε_5 , we take a linear combination of n and k , which guarantees linear independence from $\varepsilon_{1,2}$. Writing

$$\varepsilon_5 = A k + B n \quad (5.22)$$

we apply (2.14) and find that

$$0 = k \cdot \varepsilon_5 = k \cdot (A k + B n) = A k^2 + B(k \cdot n) \quad (5.23)$$

We may take

$$A = -B \frac{k \cdot n}{k^2} = B \sigma \frac{k \cdot n}{\kappa^2} \quad (5.24)$$

so that

$$\varepsilon_5 = B \left[n + \sigma \frac{k \cdot n}{\kappa^2} k \right] \quad (5.25)$$

and

$$(\varepsilon_5)^2 = B^2 \left[n^2 + 2\sigma \frac{(k \cdot n)^2}{\kappa^2} + \frac{k^2}{\kappa^2} \right] = B^2 \frac{\sigma}{\kappa^2} \left[k^2 + (k \cdot n)^2 \right] = B^2 \frac{\sigma}{\kappa^2} [k + n(k \cdot n)]^2 \quad (5.26)$$

The expression $k + n(k \cdot n)$ is just k_\perp , and its square is positive definite (expanding as a quadratic in k^0 , one sees that there are no roots). We thus find that choosing

$$B = \frac{\kappa}{|k + n(k \cdot n)|} \quad (5.27)$$

(5.26) becomes

$$(\varepsilon_5)^2 = \sigma. \quad (5.28)$$

In the simple case that $n = (1, 0, 0, 0)$, we may take as ε_1 and ε_2 the vectors

$$\varepsilon_{1,2} = \begin{bmatrix} 0 \\ \vec{\varepsilon}_{1,2} \end{bmatrix} \quad (5.29)$$

where

$$\vec{k} \cdot \vec{\varepsilon}_{1,2} = 0 \quad \vec{\varepsilon}_1 \cdot \vec{\varepsilon}_2 = 0 \quad \vec{\varepsilon}_1 \cdot \vec{\varepsilon}_1 = \vec{\varepsilon}_2 \cdot \vec{\varepsilon}_2 = 1 \quad (5.30)$$

Thus, the set $\{\vec{\varepsilon}_1, \vec{\varepsilon}_2, \hat{k} = \vec{k}/|\vec{k}|\}$ forms an orthonormal basis for the 3-space in this frame.

We find for the third polarization, ε_5 ,

$$\varepsilon_5 = \frac{1}{\kappa} \begin{bmatrix} |\vec{k}| \\ k^0 \hat{k} \end{bmatrix} \quad (5.31)$$

Notice that for $k^2 \rightarrow 0$, $\kappa \rightarrow 0$ and $\varepsilon_5 \cdot \varepsilon_5 \rightarrow 0/0$, so we will treat the lightlike case as the limit of the massive case, in which we take $\kappa \rightarrow 0$ at the end of all other computations.

In order to evaluate the commutation relations among the operators $a(k, s)$ and $a^*(k, s)$, we must first invert the Fourier expansion of $a^\mu(x, \tau)$. To do this, we write the orthogonality relations among the polarization vectors in the form

$$\varepsilon_s \cdot \varepsilon_{s'} = g_{ss'} = g(s) \delta_{ss'} \quad \text{for } s, s' = 1, 2, 5 \quad (5.32)$$

where

$$g(s) = \begin{cases} 1, & \text{if } s = s' = 1, 2 \\ \sigma, & \text{if } s = s' = 5 \end{cases} \quad (5.33)$$

so that we have from (5.14)

$$\begin{aligned} \int d^4x e^{-i(k \cdot x + \sigma \kappa \tau)} \varepsilon_s \cdot a(x, \tau) &= \\ &= \sum_{s'} \int d^4x \frac{d^4k'}{2\kappa'} \varepsilon_s \cdot \varepsilon_{s'} \left[a(k', s') e^{i(x \cdot (k' - k) + \sigma(\kappa' - \kappa)\tau)} + a^*(k', s') e^{-i(x \cdot (k + k') + \sigma(\kappa + \kappa')\tau)} \right] \\ &= \frac{(2\pi)^4}{2\kappa} g(s) \left[a(k, s) + a^*(k, s) e^{-2i\sigma\kappa\tau} \right] \end{aligned} \quad (5.34)$$

Similarly,

$$\begin{aligned} \int d^4x e^{-i(k \cdot x + \sigma \kappa \tau)} \varepsilon_s \cdot \dot{a}(x, \tau) &= \\ &= \sum_{s'} \int d^4x \frac{d^4k'}{2\kappa'} \varepsilon_s \cdot \varepsilon_{s'} (i\sigma\kappa') \\ &\quad \left[a(k', s') e^{i(x \cdot (k' - k) + \sigma(\kappa' - \kappa)\tau)} - a^*(k', s') e^{-i(x \cdot (k + k') + \sigma(\kappa + \kappa')\tau)} \right] \\ &= \frac{i\sigma}{2} (2\pi)^4 g(s) \left[a(k, s) - a^*(k, s) e^{-2i\sigma\kappa\tau} \right] \end{aligned} \quad (5.35)$$

Combining (5.34) and (5.35), we obtain

$$\begin{aligned}
i\sigma g(s)(2\pi)^4 a(k, s) &= \int d^4x e^{-i(k \cdot x + \sigma \kappa \tau)} \varepsilon_s \cdot [\dot{a}(x, \tau) + i\sigma \kappa a(x, \tau)] \\
&= \int d^4x e^{-i(k \cdot x + \sigma \kappa \tau)} \varepsilon_s \cdot [\vec{\partial}_\tau a(x, \tau) - \overleftarrow{\partial}_\tau a(x, \tau)] \\
&= \int d^4x e^{-i(k \cdot x + \sigma \kappa \tau)} \varepsilon_s \cdot [\overleftrightarrow{\partial}_\tau a(x, \tau)],
\end{aligned} \tag{5.36}$$

so that

$$a(k, s) = \frac{-i\sigma g(s)}{(2\pi)^4} \int d^4x e^{-i(k \cdot x + \sigma \kappa \tau)} \varepsilon_s \cdot [\overleftrightarrow{\partial}_\tau a(x, \tau)] \tag{5.37}$$

and

$$a^*(k, s) = \frac{i\sigma g(s)}{(2\pi)^4} \int d^4x e^{i(k \cdot x + \sigma \kappa \tau)} \varepsilon_s \cdot [\overleftrightarrow{\partial}_\tau a(x, \tau)] \tag{5.38}$$

where we have used the fact that $1/g(s) = g(s)$.

Expressions (5.37) and (5.38) permit us to evaluate the commutators

$$\begin{aligned}
[a(k, s), a^*(k', s')] &= \frac{g(s)g(s')}{(2\pi)^8} \int d^4x \int d^4x' e^{-i(k \cdot x + \sigma \kappa \tau)} e^{i(k' \cdot x' + \sigma \kappa' \tau')} \\
&\quad \left[\varepsilon_s(k) \cdot [\overleftrightarrow{\partial}_\tau a(x, \tau)], \varepsilon_{s'}(k') \cdot [\overleftrightarrow{\partial}_{\tau'} a(x', \tau')] \right],
\end{aligned} \tag{5.39}$$

where

$$\begin{aligned}
[\overleftrightarrow{\partial}_\tau a^\mu(x, \tau), \overleftrightarrow{\partial}_{\tau'} a^\nu(x', \tau')] &= \left[\left(\vec{\partial}_\tau - \overleftarrow{\partial}_\tau \right) a^\mu(x, \tau), \left(\vec{\partial}_{\tau'} - \overleftarrow{\partial}_{\tau'} \right) a^\nu(x', \tau') \right] \\
&= \left[\vec{\partial}_\tau a^\mu(x, \tau), \vec{\partial}_{\tau'} a^\nu(x', \tau') \right] + \overleftarrow{\partial}_\tau \overleftarrow{\partial}_{\tau'} [a^\mu(x, \tau), a^\nu(x', \tau')] \\
&\quad - \overleftarrow{\partial}_{\tau'} \left[\vec{\partial}_\tau a^\mu(x, \tau), a^\nu(x', \tau') \right] - \overleftarrow{\partial}_\tau \left[a^\mu(x, \tau), \vec{\partial}_{\tau'} a^\nu(x', \tau') \right]
\end{aligned} \tag{5.40}$$

so

$$\begin{aligned}
[a(k, s), a^*(k', s')] &= \frac{g(s)g(s')}{(2\pi)^8} \varepsilon_{s\mu}(k) \varepsilon_{s'\nu}(k') \int d^4x \int d^4x' e^{-i(k \cdot x + \sigma \kappa \tau)} e^{i(k' \cdot x' + \sigma \kappa' \tau')} \\
&\quad \{ [\dot{a}^\mu(x, \tau), \dot{a}^\nu(x', \tau')] + \kappa \kappa' [a^\mu(x, \tau), a^\nu(x', \tau')] \\
&\quad - i\sigma \kappa' [\dot{a}^\mu(x, \tau), a^\nu(x', \tau')] + i\sigma \kappa [a^\mu(x, \tau), \dot{a}^\nu(x', \tau')] \}.
\end{aligned} \tag{5.41}$$

Evaluating the commutators at equal- τ and using (5.6) and (5.8) to establish

$$[\dot{a}^\mu(x, \tau), a^\nu(x', \tau)] = \frac{i\sigma}{\lambda} \delta_\perp^{\mu\nu}(x - x'), \tag{5.42}$$

one obtains

$$\begin{aligned}
[a(k, s), a^*(k', s')] &= \frac{g(s)g(s')}{(2\pi)^8} \varepsilon_{s\mu}(k) \varepsilon_{s'\nu}(k') \int d^4x \int d^4x' e^{-i(k \cdot x + \sigma\kappa\tau)} e^{i(k' \cdot x' + \sigma\kappa'\tau')} \\
&\quad \left\{ -i\sigma\kappa' \left(\frac{i\sigma}{\lambda} \delta_{\perp}^{\mu\nu}(x - x') \right) + i\sigma\kappa \left(\frac{-i\sigma}{\lambda} \delta_{\perp}^{\mu\nu}(x - x') \right) \right\} \\
&= \frac{g(s)g(s')}{(2\pi)^8} \varepsilon_{s\mu}(k) \varepsilon_{s'\mu}(k') \frac{\kappa + \kappa'}{\lambda} \int d^4x e^{i[x \cdot (k' - k) + \sigma\tau(\kappa' - \kappa)]} \\
&= \frac{2\kappa}{\lambda} \frac{g(s)}{(2\pi)^4} \delta_{ss'} \delta^4(k - k')
\end{aligned} \tag{5.43}$$

where we have used $\sigma^2 = 1$, $g(s)^2 = 1$, (5.17), and (5.32).

Given the commutation relations for the operators $a(k, s)$ and $a^*(k, s)$, we may evaluate the photon propagator as the vacuum expectation value of the τ -ordered product of the fields. Thus,

$$\begin{aligned}
\langle 0 | T a_\mu(x, \tau) a_\nu(x', \tau') | 0 \rangle &= \theta(\tau - \tau') \langle 0 | a_\mu(x, \tau) a_\nu(x', \tau') | 0 \rangle \\
&\quad + \theta(\tau' - \tau) \langle 0 | a_\nu(x', \tau') a_\mu(x, \tau) | 0 \rangle
\end{aligned} \tag{5.44}$$

where we use (5.14) to expand

$$\begin{aligned}
\langle 0 | a^\mu(x, \tau) a^\nu(x', \tau') | 0 \rangle &= \sum_{s, s'} \int \frac{d^4k}{2\kappa} \frac{d^4k'}{2\kappa'} \varepsilon_s^\mu \varepsilon_{s'}^\nu \\
&\quad \langle 0 | \left[a(k, s) e^{i(k \cdot x + \sigma\kappa\tau)} + a^*(k, s) e^{-i(k \cdot x + \sigma\kappa\tau)} \right] \\
&\quad \left[a(k', s') e^{i(k' \cdot x' + \sigma\kappa'\tau')} + a^*(k', s') e^{-i(k' \cdot x' + \sigma\kappa'\tau')} \right] | 0 \rangle \\
&= \sum_{s, s'} \int \frac{d^4k}{2\kappa} \frac{d^4k'}{2\kappa'} \varepsilon_s^\mu \varepsilon_{s'}^\nu e^{i(k \cdot x + \sigma\kappa\tau)} e^{-i(k' \cdot x' + \sigma\kappa'\tau')} \langle 0 | a(k, s) a^*(k', s') | 0 \rangle \\
&= \sum_{s, s'} \int \frac{d^4k}{2\kappa} \frac{d^4k'}{2\kappa'} \varepsilon_s^\mu \varepsilon_{s'}^\nu e^{i(k \cdot x + \sigma\kappa\tau)} e^{-i(k' \cdot x' + \sigma\kappa'\tau')} \frac{2\kappa}{\lambda} \frac{g(s)}{(2\pi)^4} \delta_{ss'} \delta^4(x - x') \\
&= \sum_s \frac{g(s)}{(2\pi)^4 \lambda} \int \frac{d^4k}{2\kappa} \varepsilon_s^\mu \varepsilon_s^\nu e^{i[k \cdot (x - x') + \sigma\kappa(\tau - \tau')]}
\end{aligned} \tag{5.45}$$

so that

$$\begin{aligned}
\langle 0 | T a^\mu(x, \tau) a^\nu(x', \tau') | 0 \rangle &= \sum_s \frac{g(s)}{(2\pi)^4 \lambda} \int \frac{d^4k}{2\kappa} \varepsilon_s^\mu \varepsilon_s^\nu e^{i[k \cdot (x - x')]} \\
&\quad \left[\theta(\tau - \tau') e^{i\sigma\kappa(\tau - \tau')} + \theta(\tau' - \tau) e^{i\sigma\kappa(\tau' - \tau)} \right].
\end{aligned} \tag{5.46}$$

We recognize the integral

$$\frac{1}{2\kappa} \left[\theta(\tau - \tau') e^{i\sigma\kappa(\tau - \tau')} \theta(\tau' - \tau) e^{i\sigma\kappa(\tau' - \tau)} \right] = -\frac{i}{2\pi} \int dK \frac{e^{i\sigma K(\tau - \tau')}}{k^2 + \sigma K^2 - i\epsilon} \tag{5.47}$$

which we may use to put the Green's function (the vacuum expectation value of the τ -ordered product of the fields) in the form of the Feynman propagator for the five-dimensional field:

$$\begin{aligned} d^{\mu\nu}(x-x', \tau-\tau') &= i\langle 0|T a^\mu(x, \tau) a^\nu(x', \tau')|0\rangle \\ &= \int \frac{d^4k d\kappa}{(2\pi)^5} \sum_s \frac{g(s)}{\lambda} \varepsilon_s^\mu \varepsilon_s^\nu \frac{e^{i[k \cdot (x-x') + \sigma \kappa(\tau-\tau')]} }{k^2 + \sigma \kappa^2 - i\epsilon} \end{aligned} \quad (5.48)$$

In order to evaluate the sum over polarizations in (5.48), we must consider the cases $\sigma = \pm 1$ separately. For the case that $\sigma = 1$, we must satisfy $k^2 = -\kappa^2 < 0$, and we may take

$$k = \lim_{\alpha \rightarrow 0} (\sqrt{\kappa^2 + \alpha^2}, 0, 0, \alpha) = (\kappa, 0, 0, 0). \quad (5.49)$$

Choosing

$$n = (1, 0, 0, 0) \quad \varepsilon_1 = (0, 1, 0, 0) \quad \varepsilon_2 = (0, 0, 1, 0) \quad (5.50)$$

we find from (5.31) and (5.49) that

$$\varepsilon_5 = (|\vec{k}|, k^0 \hat{k}) \frac{1}{\sqrt{-k^2}} = \lim_{\alpha \rightarrow 0} \frac{1}{\sqrt{\kappa^2 + \alpha^2}} (\alpha, 0, 0, \sqrt{\kappa^2 + \alpha^2}) = (0, 0, 0, 1). \quad (5.51)$$

The completeness relation may then be written in the form

$$g^{\mu\nu} = -\frac{1}{\kappa^2} k^\mu k^\nu + (\varepsilon_1)^\mu (\varepsilon_1)^\nu + (\varepsilon_2)^\mu (\varepsilon_2)^\nu + (\varepsilon_5)^\mu (\varepsilon_5)^\nu \quad (5.52)$$

which we may rearrange to write

$$\sum_{s=1,2,5} g(s) (\varepsilon_s)^\mu (\varepsilon_s)^\nu = \sum_{s=1,2,5} (\varepsilon_s)^\mu (\varepsilon_s)^\nu = g^{\mu\nu} + \frac{1}{\kappa^2} k^\mu k^\nu = g^{\mu\nu} - \frac{1}{k^2} k^\mu k^\nu \quad (5.53)$$

where we have used $g(1, 2) = 1$ and $g(5) = \sigma = 1$.

For the case of $\sigma = -1$, we have $k^2 = \kappa^2 > 0$ so we may take

$$k = (0, 0, 0, \kappa). \quad (5.54)$$

and using (5.31) we find that

$$\varepsilon_5 = (|\vec{k}|, k^0 \hat{k}) \frac{1}{\sqrt{-k^2}} = (1, 0, 0, 0). \quad (5.55)$$

In this case the completeness relation is

$$g^{\mu\nu} = -(\varepsilon_5)^\mu (\varepsilon_5)^\nu + (\varepsilon_1)^\mu (\varepsilon_1)^\nu + (\varepsilon_2)^\mu (\varepsilon_2)^\nu + \frac{1}{\kappa^2} k^\mu k^\nu \quad (5.56)$$

and it may be rearranged as

$$\sum_{s=1,2,5} g(s)(\varepsilon_s)^\mu(\varepsilon_s)^\nu = \sum_{s=1,2} (\varepsilon_s)^\mu(\varepsilon_s)^\nu - (\varepsilon_5)^\mu(\varepsilon_5)^\nu = g^{\mu\nu} - \frac{1}{\kappa^2}k^\mu k^\nu = g^{\mu\nu} - \frac{1}{k^2}k^\mu k^\nu \quad (5.57)$$

where we have used $g(1, 2) = 1$ and $g(5) = \sigma = -1$. Notice that (5.57) and (5.53) are identical and are consistent with the transversality requirements on the polarization states. Using this expression and observing that the sum over polarization states gives the projection operator $\mathcal{P}^{\mu\nu}(k)$ defined in (5.2), equation (5.7) may be verified explicitly.

To evaluate the Hamiltonian in the momentum representation,

$$\begin{aligned} \frac{\lambda}{4} f_{\mu\nu} f^{\mu\nu} &= \frac{\lambda}{4} (\partial_\mu a_\nu - \partial_\nu a_\mu)(\partial^\mu a^\nu - \partial^\nu a^\mu) \\ &= \frac{\lambda}{2} [(\partial_\mu a_\nu)(\partial^\mu a^\nu) - (\partial_\nu a_\mu)(\partial^\mu a^\nu)] \\ &= \frac{\lambda}{2} [\partial^\mu (a^\nu \partial_\mu a_\nu) - a_\nu \square a^\nu - \partial^\mu (a^\nu \partial_\nu a_\mu) - a^\nu \partial_\nu (\partial^\mu a_\mu)] \\ &= -\frac{\lambda}{2} a_\nu \square a^\nu + \text{divergence} \end{aligned} \quad (5.58)$$

and

$$\begin{aligned} -\frac{\lambda\sigma}{2} \epsilon^\mu \epsilon_\mu &= -\frac{\lambda\sigma}{2} (\sigma \partial_\tau a_\mu)(\sigma \partial_\tau a^\mu) \\ &= -\frac{\lambda\sigma}{2} [\partial_\tau (a^\mu \partial_\tau a_\mu) - a_\mu \partial_\tau^2 a^\mu] \\ &= \frac{\lambda\sigma}{2} a_\mu \partial_\tau^2 a^\mu. \end{aligned} \quad (5.59)$$

So the Hamiltonian becomes

$$K = \frac{\lambda}{2} \int d^4x a_\mu [-\square + \sigma \partial_\tau^2] a^\mu = \lambda\sigma \int d^4x a_\mu \partial_\tau^2 a^\mu \quad (5.60)$$

where we have used the wave equation (5.13). Using the momentum expansion (5.14) and the commutation relations (5.43) we find for the normal ordered Hamiltonian

$$:K: = \lambda\sigma \int d^4x \sum_{s,s'} \int \frac{d^4k d^4k'}{4\kappa\kappa'} (\varepsilon_s)^\mu (\varepsilon_{s'})_\mu (i\sigma\kappa')^2 \quad (5.61)$$

$$\begin{aligned} &: \left\{ a(k, s) a(k', s') e^{i[(k+k') \cdot x + \sigma(\kappa+\kappa')\tau]} + a^*(k, s) a^*(k', s') e^{-i[(k+k') \cdot x + \sigma(\kappa+\kappa')\tau]} \right. \\ &\quad \left. + a^*(k, s) a(k', s') e^{-i[(k-k') \cdot x + \sigma(\kappa-\kappa')\tau]} + a^*(k', s') a(k, s) e^{i[(k-k') \cdot x + \sigma(\kappa-\kappa')\tau]} \right\} : \\ &= -\lambda\sigma(2\pi)^4 \sum_{s,s'} \int \frac{d^4k}{4\kappa^2} \kappa^2 (\varepsilon_s)^\mu (\varepsilon_{s'})_\mu \\ &: \left\{ a(k, s) a(-k, s') e^{2i\sigma\kappa\tau} + a^*(k, s) a^*(-k, s') e^{-2i\sigma\kappa\tau} \right. \\ &\quad \left. + a^*(k, s) a(k, s') + a^*(k, s') a(k, s) \right\} : \end{aligned} \quad (5.62)$$

By replacing $k \rightarrow -k$ in (5.14), we see that we may identify

$$a(-k, s)e^{i\sigma\kappa\tau} = a^*(k, s)e^{-i\sigma\kappa\tau} \quad a^*(-k, s)e^{-i\sigma\kappa\tau} = a(k, s)e^{i\sigma\kappa\tau}. \quad (5.63)$$

Using (5.63) in (5.62), we obtain

$$\begin{aligned} :K: &= -\lambda\sigma(2\pi)^4 \sum_{s,s'} \int \frac{d^4k}{4\kappa^2} \kappa^2 (\varepsilon_s)^\mu (\varepsilon_{s'})_\mu 2 : \{a^*(k, s)a(k, s') + a^*(k, s')a(k, s)\} : \\ &= -\lambda\sigma(2\pi)^4 \sum_{s,s'} \int \frac{d^4k}{4\kappa^2} 2\kappa^2 [g(s)\delta_{ss'}] \{a^*(k, s)a(k, s') + a^*(k, s')a(k, s)\} \\ &= -\lambda\sigma(2\pi)^4 \sum_s g(s) \int d^4k a^*(k, s)a(k, s) \end{aligned} \quad (5.64)$$

where we have used (5.32).

6 Perturbation Theory

We define $|\psi\rangle$ to be a τ -independent Heisenberg state and

$$|\psi(\tau)\rangle = U(\tau)|\psi(0)\rangle = U(\tau)|\psi\rangle \quad (6.1)$$

to be the corresponding Schrödinger state. We require that for all τ ,

$$\langle\psi(\tau)|\psi(\tau)\rangle = \langle\psi|\overline{U}U|\psi\rangle = \langle\psi|\psi\rangle, \quad (6.2)$$

so that

$$\overline{U}(\tau)U(\tau) = 1 \quad (6.3)$$

Since U is unitary and $U(0) = 1$, it may be expressed in terms of a Hermitian generator $K(\tau)$, such that

$$U(\tau) = e^{-iK(\tau)} \quad i\partial_\tau U(\tau)|_{\tau=0} = K'(0). \quad (6.4)$$

But also,

$$U(\tau + \tau') = U(\tau)U(\tau') \quad (6.5)$$

so that

$$\begin{aligned} i\partial_\tau U(\tau) &= \lim_{\tau' \rightarrow 0} i \frac{U(\tau + \tau') - U(\tau)}{\tau'} \\ &= \lim_{\tau' \rightarrow 0} i \frac{U(\tau')U(\tau) - U(0)U(\tau)}{\tau'} \end{aligned}$$

$$\begin{aligned}
&= \lim_{\tau' \rightarrow 0} i \frac{U(\tau') - U(0)}{\tau'} U(\tau) \\
&= i \partial_\tau U(\tau)|_{\tau=0} U(\tau) \\
&= K'(0) U(\tau).
\end{aligned} \tag{6.6}$$

Therefore, we find

$$K(\tau) = K\tau \tag{6.7}$$

which says that $|\psi(\tau)\rangle$ satisfies the Schrödinger equation. For a τ -independent operator A_S in the Schrödinger picture,

$$A(\tau) = \langle \psi(\tau) | A_S | \psi(\tau) \rangle = \langle \psi | \bar{U}(\tau) A_S U(\tau) | \psi \rangle = \langle \psi | A_H(\tau) | \psi \rangle \tag{6.8}$$

where the Heisenberg operator $A_H(\tau)$ satisfies

$$i \partial_\tau A_H(\tau) = i \partial_\tau [e^{iK\tau} A_S e^{-iK\tau}] = [A_H(\tau), K] \tag{6.9}$$

Suppose that

$$K = K_0 + K' \tag{6.10}$$

where K_0 is the Hamiltonian of a free event. The field operator ϕ_H for the interacting event must satisfy (6.9) and must have a corresponding Schrödinger operator such that

$$\phi_H(x, \tau) = e^{iK\tau} \phi_S(x) e^{-iK\tau} \tag{6.11}$$

The interaction picture operator is defined by [15]

$$\phi_I(x, \tau) = e^{iK_0\tau} \phi_S(x) e^{-iK_0\tau} = e^{iK_0\tau} e^{-iK\tau} \phi_H(x, \tau) e^{iK\tau} e^{-iK_0\tau} \tag{6.12}$$

so that in the limit that $K' \rightarrow 0$, $\phi_I \rightarrow \phi_H$. Since $\phi_S(x)$ is τ -independent, ϕ_I satisfies

$$i \partial_\tau \phi_I(x, \tau) = [\phi_I(x, \tau), K_0] \tag{6.13}$$

while the states of the interaction picture are

$$|\phi(\tau)\rangle_I = e^{iK_0\tau} |\phi(\tau)\rangle_S = e^{iK_0\tau} e^{-iK\tau} |\phi\rangle_H \tag{6.14}$$

and go over to the Heisenberg states in the absence of interaction. From (6.14) we see that

$$\begin{aligned}
i \partial_\tau |\phi(\tau)\rangle_I &= e^{iK_0\tau} K e^{-iK\tau} |\phi(\tau)\rangle_H - e^{iK_0\tau} K_0 e^{-iK\tau} |\phi(\tau)\rangle_H \\
&= e^{iK_0\tau} [K - K_0] e^{-iK\tau} |\phi(\tau)\rangle_H \\
&= e^{iK_0\tau} K' e^{-iK_0\tau} e^{iK_0\tau} e^{-iK\tau} |\phi(\tau)\rangle_H \\
&= e^{iK_0\tau} K' e^{-iK_0\tau} |\phi(\tau)\rangle_I \\
&= K_I(\tau) |\phi(\tau)\rangle_I.
\end{aligned} \tag{6.15}$$

Equation (6.15) has the formal solution

$$|\phi(\tau)\rangle_I = V(\tau, \tau')|\phi(\tau')\rangle_I = \text{T}e^{-i \int_{\tau'}^{\tau} d\tau'' K_I(\tau'')} |\phi(\tau')\rangle_I \quad (6.16)$$

in which the symbol T refers to τ -ordering. The S-matrix is given by

$$S = \lim_{\substack{\tau' \rightarrow -\infty \\ \tau \rightarrow \infty}} \text{T}e^{-i \int_{\tau'}^{\tau} d\tau'' K_I(\tau'')} = \text{T}e^{-i \int d^4x d\tau K_I} \quad (6.17)$$

Scattering amplitudes may be computed from the interaction picture states as

$$W_{i \rightarrow f} = \langle f | S | i \rangle. \quad (6.18)$$

Since the initial and final states are defined when there is no interaction, the asymptotic interaction picture states are identical with the asymptotic Heisenberg states. Moreover, since the Heisenberg picture Hamiltonian is formed of bilinear combinations of the fields without τ -derivatives, the interacting Hamiltonian in (6.17) has the same form as the Heisenberg picture Hamiltonian.

Reduction Formulas

In order to use Wick's theorem for perturbation expansions of the S-matrix, [40], we must adapt the LSZ reduction formulas [40] to the present theory. Consider the amplitude

$$\langle k' \text{ out} | k \text{ in} \rangle = \langle k' \text{ out} | b_{in}^*(k) | 0 \rangle \quad (6.19)$$

From (4.6) we find that

$$b^*(k) = \int d^4x e^{i(k \cdot x - \kappa \tau)} \psi^*(x, \tau) \quad b(k) = \int d^4x e^{i(k \cdot x - \kappa \tau)} \psi(x, \tau) \quad (6.20)$$

which we may insert in (6.19) to obtain

$$\langle k' \text{ out} | k \text{ in} \rangle = \int d^4x \langle k' \text{ out} | \psi_{in}^*(x, \tau) | 0 \rangle e^{i(k \cdot x - \kappa \tau)} \quad (6.21)$$

which is valid for arbitrary τ_{in} . Now, consider the integral

$$\begin{aligned} \int_{\tau_i}^{\tau_f} d\tau \int d^4x \partial_{\tau} \left\{ \langle k' \text{ out} | \psi^*(x, \tau) | 0 \rangle e^{i(k \cdot x - \kappa \tau)} \right\} &= \\ &= \int d^4x e^{i(k \cdot x - \kappa \tau)} \langle k' \text{ out} | \psi^*(x, \tau_f) - \psi^*(x, \tau_i) | 0 \rangle \\ &= \langle k' \text{ out} | b_{out}^*(k) | 0 \rangle - \int d^4x e^{i(k \cdot x - \kappa \tau)} \langle k' \text{ out} | \psi^*(x, \tau_i) | 0 \rangle, \end{aligned} \quad (6.22)$$

which we may rearrange as

$$\begin{aligned}
\langle k' \text{ out} | k \text{ in} \rangle &= - \int d^4x d\tau \partial_\tau \left\{ e^{i(k \cdot x - \kappa \tau)} \langle k' \text{ out} | \psi^*(x, \tau) | 0 \rangle \right\} + \langle k' \text{ out} | b_{out}^*(k) | 0 \rangle \\
&= - \int d^4x d\tau e^{i(k \cdot x - \kappa \tau)} \left\{ -i\kappa + \partial_\tau \right\} \langle k' \text{ out} | \psi^*(x, \tau) | 0 \rangle + \langle k' \text{ out} | b_{out}^*(k) | 0 \rangle \\
&= - \int d^4x d\tau e^{i(k \cdot x - \kappa \tau)} \left\{ i \frac{k^2}{2M} + \partial_\tau \right\} \langle k' \text{ out} | \psi^*(x, \tau) | 0 \rangle + \langle k' \text{ out} | b_{out}^*(k) | 0 \rangle \\
&= - \int d^4x d\tau e^{i(k \cdot x - \kappa \tau)} \left\{ \frac{i}{2M} \square + \partial_\tau \right\} \langle k' \text{ out} | \psi^*(x, \tau) | 0 \rangle + \langle k' \text{ out} | b_{out}^*(k) | 0 \rangle \\
&= i \int d^4x d\tau e^{i(k \cdot x - \kappa \tau)} \left\{ i \partial_\tau - \frac{1}{2M} \square \right\} \langle k' \text{ out} | \psi^*(x, \tau) | 0 \rangle + \langle k' \text{ out} | b_{out}^*(k) | 0 \rangle
\end{aligned} \tag{6.23}$$

where we performed two integrations by parts in the second to last step. The last term is a “disconnected term”; it corresponds to the non-scattering path and makes no contribution to the scattering amplitude. Now consider the integral

$$\begin{aligned}
\int_{\tau_i}^{\tau_f} d\tau' \int d^4x' \partial_{\tau'} \left[\langle 0 | T \psi(x', \tau') \psi^*(x, \tau) | 0 \rangle e^{-i(k' \cdot x' - \kappa' \tau')} \right] &= \\
&= \int d^4x' e^{-i(k' \cdot x' - \kappa' \tau')} \langle 0 | T \psi(x', \tau') \psi^*(x, \tau) | 0 \rangle \Big|_{\tau_i}^{\tau_f} \\
&= \langle 0 | b_{out}(k') \psi^*(x, \tau) | 0 \rangle - \langle 0 | \psi^*(x, \tau) b_{in}(k') | 0 \rangle
\end{aligned} \tag{6.24}$$

in which the second term in (6.24) vanishes. For the second term in (6.23), we use (6.24) to similarly expand as

$$\begin{aligned}
\langle k' \text{ out} | \psi^*(x, \tau) | 0 \rangle &= \langle 0 | b_{out}(k') \psi^*(x, \tau) | 0 \rangle \\
&= \int d^4x' e^{i(k' \cdot x' - \kappa' \tau')} \langle 0 | \psi_{out}(x', \tau') \psi^*(x, \tau) | 0 \rangle \\
&= \int d^4x' d\tau' \partial_{\tau'} \left\{ e^{-i(k' \cdot x' - \kappa' \tau')} \langle 0 | T \psi(x', \tau') \psi^*(x, \tau) | 0 \rangle \right\} + \\
&\quad \langle 0 | \psi^*(x, \tau) b_{in}(k') | 0 \rangle
\end{aligned} \tag{6.25}$$

and again the second term vanishes. Finally, we arrive at

$$\begin{aligned}
\langle k' \text{ out} | k \text{ in} \rangle &= i^2 \int d^4x d\tau d^4x' d\tau' e^{i(k \cdot x - \kappa \tau)} e^{-i(k' \cdot x' - \kappa' \tau')} \\
&\quad \left[i \partial_{\tau'} + \frac{1}{2M} \square' \right] \left[i \partial_\tau - \frac{1}{2M} \square \right] \langle 0 | T \psi(x', \tau') \psi^*(x, \tau) | 0 \rangle
\end{aligned} \tag{6.26}$$

By continuing this process, we can write

$$\langle k'_1 \cdots k'_n \text{ out} | k_1 \cdots k_m \text{ in} \rangle = i^{n+m} \int d^5x_1 \cdots d^5x_m d^5x'_1 \cdots d^5x'_n$$

$$\begin{aligned}
& e^{i(k_1 \cdot x_1 + \dots + k_m \cdot x_m - \kappa_1 \tau_1 - \dots - \kappa_m \tau_m)} e^{-i(k'_1 \cdot x'_1 + \dots + k'_n \cdot x'_n - \kappa'_1 \tau'_1 - \dots - \kappa'_n \tau'_n)} \\
& [i\partial_{\tau'_1} + \frac{1}{2M}\square'_1] \dots [i\partial_{\tau'_n} + \frac{1}{2M}\square'_n] [i\partial_{\tau_1} - \frac{1}{2M}\square_1] \dots [i\partial_{\tau_m} - \frac{1}{2M}\square_m] \\
& \langle 0 | T \psi(x'_1, \tau'_1) \dots \psi(x'_n, \tau'_n) \psi^*(x_1, \tau_1) \dots \psi^*(x_m, \tau_m) | 0 \rangle
\end{aligned} \tag{6.27}$$

where we denote $d^5x = d^4x d\tau$.

For the gauge field, the reduction formula follows closely the development for the usual Maxwell case. We begin with

$$\begin{aligned}
\langle \beta, k, s \text{ out} | \alpha \text{ in} \rangle &= \langle \beta \text{ out} | a(k, s) | \alpha \text{ in} \rangle \\
&= \frac{-i\sigma g(s)}{(2\pi)^4} \int d^4x e^{-i(k \cdot x + \sigma \kappa \tau)} \langle \beta \text{ out} | \varepsilon_s \cdot \overleftrightarrow{\partial}_\tau a(x, \tau) | \alpha \text{ in} \rangle
\end{aligned} \tag{6.28}$$

where α and β are the other quantum numbers for the states. We may again use

$$\int_{\tau_i}^{\tau_f} d\tau \int d^4x \partial_\tau g(x, \tau) = \int d^4x [g(x, \tau_f) - g(x, \tau_i)] \tag{6.29}$$

to write

$$\begin{aligned}
& \int_{\tau_i}^{\tau_f} d\tau \int d^4x \partial_\tau \frac{-i\sigma g(s)}{(2\pi)^4} e^{-i(k \cdot x + \sigma \kappa \tau)} \langle \beta \text{ out} | \varepsilon_s \cdot \overleftrightarrow{\partial}_\tau a(x, \tau) | \alpha \text{ in} \rangle = \\
&= \frac{-i\sigma g(s)}{(2\pi)^4} \int d^4x e^{-i(k \cdot x + \sigma \kappa \tau)} \left\{ \langle \beta \text{ out} | \varepsilon_s \cdot \overleftrightarrow{\partial}_\tau a(x, \tau_f) | \alpha \text{ in} \rangle \right. \\
&\quad \left. - \langle \beta \text{ out} | \varepsilon_s \cdot \overleftrightarrow{\partial}_\tau a(x, \tau_i) | \alpha \text{ in} \rangle \right\} \\
&= \frac{-i\sigma g(s)}{(2\pi)^4} \int d^4x e^{-i(k \cdot x + \sigma \kappa \tau)} \langle \beta \text{ out} | \varepsilon_s \cdot \overleftrightarrow{\partial}_\tau a(x, \tau_f) | \alpha \text{ in} \rangle
\end{aligned} \tag{6.30}$$

where we have used the fact that $a(x, \tau_i) | 0 \text{ in} \rangle = 0$. By expanding $\overleftrightarrow{\partial}_\tau$, using the wave equation (5.13), and performing two integrations by parts, we arrive at

$$\langle \beta, k, s \text{ out} | \alpha \text{ in} \rangle = \frac{-ig(s)}{(2\pi)^4} \int d^5x e^{-i(k \cdot x + \sigma \kappa \tau)} [\square + \sigma \partial_\tau^2] \langle \beta \text{ out} | \varepsilon_s \cdot a(x, \tau) | \alpha \text{ in} \rangle \tag{6.31}$$

A second application of this procedure enables us to write

$$\begin{aligned}
\langle \beta, k, s \text{ out} | \alpha, k', s' \text{ in} \rangle &= \frac{(-i)^2 g(s) g(s')}{(2\pi)^8} \int d^5x d^5x' e^{-i(k \cdot x + \sigma \kappa \tau)} e^{i(k' \cdot x' + \sigma \kappa' \tau')} \\
&\quad [\square + \sigma \partial_\tau^2] [\square' + \sigma \partial_{\tau'}^2] \langle \beta \text{ out} | \varepsilon_s \cdot a(x, \tau) \varepsilon_{s'} \cdot a(x', \tau') | \alpha \text{ in} \rangle
\end{aligned} \tag{6.32}$$

Feynman Rules

In order to write the Feynman rules for the interacting off-shell theory, may use the general expression for Green's functions, which may be derived from the path integral [41] or by operator methods [40],

$$G^{(n)}(x_1, \tau_1, \dots, x_n, \tau_n) = \langle 0 | T \phi(x_1, \tau_1), \dots, \phi(x_n, \tau_n) e^{i \int d^4 y d\tau \mathcal{L}_{\text{int}}} | 0 \rangle \quad (6.33)$$

in which vacuum-vacuum diagrams are excluded from (6.33), and ϕ represents the non-interacting free fields of the theory, which permits us to use Wick's theorem and the free propagators (4.15) and (5.48). In (6.33), we adopt the conventional notation for Green's functions in field theory; in relation to the non-interaction propagators calculated above, these are

$$\begin{aligned} G^{(2)}(x, \tau) &= \langle 0 | T \psi(x, \tau) \psi^*(0) | 0 \rangle_{\text{tree}} = -iG(x, \tau) \\ d_{\mu\nu}^{(2)}(x, \tau) &= \langle 0 | T a_\mu(x, \tau) a_\nu(0) | 0 \rangle_{\text{tree}} = -i d_{\mu\nu}(x, \tau) \end{aligned} \quad (6.34)$$

where $G(x, \tau)$ and $d_{\mu\nu}(x, \tau)$ are defined in (4.11) and (5.48).

Using (2.31) for the interacting Hamiltonian, we have

$$\mathcal{L}_{\text{int}} = -\mathcal{K}_{\text{interaction}} = -\frac{ie_0}{2M} a_\mu (\psi^* \partial^\mu \psi - (\partial^\mu \psi^*) \psi) - \frac{e_0^2}{2M} a_\mu a^\mu |\psi|^2. \quad (6.35)$$

The two terms in (6.35) correspond to the two basic diagrams



and we may calculate the vertex factors separately. For the first diagram, we have

$$G_\mu^{(3)}(x_1, \tau_1, x_2, \tau_2, x_3, \tau_3) = \langle 0 | T \psi^*(x_1, \tau_1) \psi(x_2, \tau_2) a_\mu(x_3, \tau_3) e^{i \int d^4 y d\tau e_0 a_\mu(y, \tau) j^\mu(y, \tau)} | 0 \rangle_{\text{tree}} \quad (6.36)$$

where only the tree-level diagram is considered in (6.36) and j^μ is the vector part of the free matter field current (1.25),

$$j^\mu = \frac{-i}{2M} (\psi^* \partial^\mu \psi - (\partial^\mu \psi^*) \psi) \quad (6.37)$$

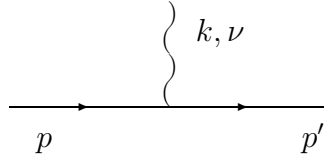
Expanding the exponential in (6.36), we find that the tree-level term is given by

$$\begin{aligned}
G_\mu^{(3)}(x_1, \tau_1, x_2, \tau_2, x_3, \tau_3) &= \\
&= \langle 0 | T \psi^*(x_1, \tau_1) \psi(x_2, \tau_2) a_\mu(x_3, \tau_3) \left\{ i e_0 \int d^4 y d\tau a_\mu(y, \tau) j^\mu(y, \tau) \right\} | 0 \rangle \\
&= \frac{e_0}{2M} \langle 0 | T \psi^*(x_1, \tau_1) \psi(x_2, \tau_2) a_\mu(x_3, \tau_3) \int d^4 y d\tau a^\nu(y, \tau) \psi^*(y, \tau) \overleftrightarrow{\partial}_{y^\nu} \psi(y, \tau) | 0 \rangle \\
&= \frac{e_0}{2M} \int d^4 y d\tau \langle 0 | T a_\mu(x_3, \tau_3) a^\nu(y, \tau) | 0 \rangle \langle 0 | T \psi^*(x_1, \tau_1) \psi(x_2, \tau_2) \psi^*(y, \tau) \overleftrightarrow{\partial}_{y^\nu} \psi(y, \tau) | 0 \rangle \\
&= \frac{e_0}{2M} \int d^4 y d\tau \langle 0 | T a_\mu(x_3, \tau_3) a^\nu(y, \tau) | 0 \rangle \\
&\quad [\langle 0 | T \psi(x_2, \tau_2) \psi^*(y, \tau) | 0 \rangle \overleftrightarrow{\partial}_{y^\nu} \langle 0 | T \psi^*(x_1, \tau_1) \psi(y, \tau) | 0 \rangle] \\
&= \frac{e_0}{2M} \int d^4 y d\tau [-i d_\mu^\nu(x_3 - y, \tau - \tau_3)] \\
&\quad [(-iG(x_2 - y, \tau_2 - \tau)) \overleftrightarrow{\partial}_{y^\nu} (-iG(y - x_1, \tau - \tau_1))]
\end{aligned} \tag{6.38}$$

We now use the Fourier expansions for the free propagators to write

$$\begin{aligned}
G_\mu^{(3)}(x_1, \tau_1, x_2, \tau_2, x_3, \tau_3) &= \\
&= \frac{i e_0}{2M} \int d^4 y d\tau \frac{d^5 k d^5 p d^5 p'}{(2\pi)^{15} \lambda} \mathcal{P}_\mu^\nu(k) \frac{e^{i[k \cdot (x_3 - y) + \sigma \kappa (\tau_3 - \tau)]}}{k^2 + \sigma \kappa^2 - i\epsilon} \\
&\quad \left[\frac{e^{i(p' \cdot (x_2 - y) - P'(\tau_2 - \tau))}}{\frac{1}{2M}(p')^2 - P' - i\epsilon} \overleftrightarrow{\partial}_{y^\nu} \frac{e^{i(p \cdot (y - x_1) - P(\tau - \tau_1))}}{\frac{1}{2M}p^2 - P - i\epsilon} \right] \\
&= \frac{i e_0}{2M} \int d^4 y d\tau \frac{d^5 k d^5 p d^5 p'}{(2\pi)^{15} \lambda} \mathcal{P}_\mu^\nu(k) i(p + p')^\nu e^{i y \cdot [p - p' - k] - i \tau [P - P' + \sigma \kappa]} \\
&\quad \frac{e^{i(k \cdot x_3 + \sigma \kappa \tau_3)}}{k^2 + \sigma \kappa^2 - i\epsilon} \frac{e^{i(p' \cdot x_2) - P' \tau_2}}{\frac{1}{2M}(p')^2 - P' - i\epsilon} \frac{e^{-i(p \cdot x_1) - P \tau_1}}{\frac{1}{2M}p^2 - P - i\epsilon} \\
&= \frac{-e_0}{2M} \int \frac{d^5 k d^5 p d^5 p'}{(2\pi)^{15} \lambda} \mathcal{P}_\mu^\nu(k) (p + p')^\nu (2\pi)^5 \delta^4(p - p' - k) \delta(P - P' + \sigma \kappa) \\
&\quad \frac{e^{i(k \cdot x_3 + \sigma \kappa \tau_3)}}{k^2 + \sigma \kappa^2 - i\epsilon} \frac{e^{i(p' \cdot x_2) - P' \tau_2}}{\frac{1}{2M}(p')^2 - P' - i\epsilon} \frac{e^{-i(p \cdot x_1) - P \tau_1}}{\frac{1}{2M}p^2 - P - i\epsilon}
\end{aligned} \tag{6.39}$$

Transforming to the momentum space Green's function, with diagram



we find

$$G_\mu^{(3)}(p, p', k) = \frac{e_0}{2M} \mathcal{P}_\mu^\nu(k) i(p+p')_\nu (2\pi)^5 \delta^4(p-p'-k) \delta(P-P'+\sigma\kappa) \frac{1}{\lambda} \frac{-i}{k^2 + \sigma\kappa^2 - i\epsilon} \frac{-i}{\frac{1}{2M}(p')^2 - P' - i\epsilon} \frac{-i}{\frac{1}{2M}p^2 - P - i\epsilon} \quad (6.40)$$

which we identify as the product of the three propagators and the vertex factor for the interaction

$$\frac{e_0}{2M} i(p+p')^\nu (2\pi)^5 \delta^4(p-p'-k) \delta(P-P'+\sigma\kappa) \quad (6.41)$$

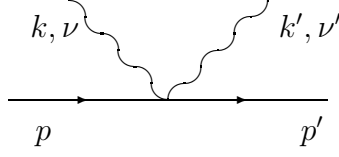
in which

$$\kappa = \sqrt{-\sigma k^2} \quad P = \frac{p^2}{2M} \quad P' = \frac{p'^2}{2M} \quad (6.42)$$

For the second diagram, we calculate

$$\begin{aligned} G_{\mu\nu}^{(4)}(x_1, \tau_1, x_2, \tau_2, x_3, \tau_3, x_4, \tau_4) &= \\ &= \langle 0 | T \psi^*(x_1, \tau_1) \psi(x_2, \tau_2) a_\mu(x_3, \tau_3) a_\nu(x_4, \tau_4) e^{-i \int d^4 y d\tau \frac{e_0^2}{2M} a_\rho a^\rho |\psi|^2} | 0 \rangle_{\text{tree}} \\ &= \frac{-ie_0^2}{2M} \int d^4 y d\tau \langle 0 | T \psi^*(x_1, \tau_1) \psi(x_2, \tau_2) a_\mu(x_3, \tau_3) a_\nu(x_4, \tau_4) \\ &\quad a_\rho(y, \tau) a^\rho(y, \tau) \psi^*(y, \tau) \psi(y, \tau) | 0 \rangle \\ &= \frac{-ie_0^2}{2M} \int d^4 y d\tau \langle 0 | T \psi^*(x_1, \tau_1) \psi(x_2, \tau_2) \psi^*(y, \tau) \psi(y, \tau) | 0 \rangle \\ &\quad \langle 0 | T a_\mu(x_3, \tau_3) a_\nu(x_4, \tau_4) a_\rho(y, \tau) a^\rho(y, \tau) | 0 \rangle \\ &= \frac{-ie_0^2}{2M} \int d^4 y d\tau \langle 0 | T \psi^*(x_1, \tau_1) \psi(y, \tau) | 0 \rangle \langle 0 | T \psi(x_2, \tau_2) \psi^*(y, \tau) | 0 \rangle \\ &\quad 2 \left\{ \langle 0 | T a_\mu(x_3, \tau_3) a_\rho(y, \tau) | 0 \rangle \langle 0 | T a_\nu(x_4, \tau_4) a^\rho(y, \tau) | 0 \rangle \right\} \\ &= \frac{-ie_0^2}{M} \int d^4 y d\tau \frac{d^5 k d^5 k' d^5 p d^5 p'}{(2\pi)^{20} \lambda^2} \mathcal{P}_\mu^\rho(k) \frac{e^{i[k \cdot (y-x_3) + \sigma\kappa(\tau-\tau_3)]}}{k^2 + \sigma\kappa^2 - i\epsilon} \mathcal{P}_{\rho\nu}(k') \times \\ &\quad \times \frac{e^{i[k' \cdot (x_4-y) + \sigma\kappa'(\tau_4-\tau)]}}{(k')^2 + \sigma(\kappa')^2 - i\epsilon} \frac{e^{i[p' \cdot (x_2-y) - P'(\tau_2-\tau)]}}{\frac{1}{2M}(p')^2 - P' - i\epsilon} \frac{e^{i[p \cdot (y-x_1) - P(\tau-\tau_1)]}}{\frac{1}{2M}p^2 - P - i\epsilon} \\ &= \frac{-ie_0^2}{M\lambda^2} \int \frac{d^5 k d^5 k' d^5 p d^5 p'}{(2\pi)^{20}} (2\pi)^5 \delta^4(k - k' - p' + p) \delta(\sigma\kappa - \sigma\kappa' + P' - P) \mathcal{P}_\mu^\rho(k) \\ &\quad \mathcal{P}_{\rho\nu}(k') e^{-i[k \cdot x_3 + \sigma\kappa\tau_3]} e^{i[k' \cdot x_4 + \sigma\kappa'\tau_4]} e^{i[p' \cdot x_2 - P'\tau_2]} e^{-i[p \cdot x_1 - P\tau_1]} \\ &\quad \frac{-i}{k^2 + \sigma\kappa^2 - i\epsilon} \frac{-i}{(k')^2 + \sigma(\kappa')^2 - i\epsilon} \frac{-i}{\frac{1}{2M}(p')^2 - P' - i\epsilon} \frac{-i}{\frac{1}{2M}p^2 - P - i\epsilon} \end{aligned} \quad (6.43)$$

Notice that a factor of 2 appears in the fourth line, because there are two ways to contract the photon operators. Transforming to the momentum space Green's function, with diagram



we find

$$\begin{aligned}
G_{\mu\nu}^{(4)}(p, p', k, k') &= \\
&= \frac{-ie_0^2}{M\lambda^2} (2\pi)^5 \delta^4(k - k' - p' + p) \delta(\sigma\kappa - \sigma\kappa' + P' - P) \mathcal{P}_\mu^\rho(k) \mathcal{P}_{\rho\nu}(k') \\
&\quad \frac{-i}{k^2 + \sigma\kappa^2 - i\epsilon} \frac{-i}{(k')^2 + \sigma(\kappa')^2 - i\epsilon} \frac{-i}{\frac{1}{2M}(p')^2 - P' - i\epsilon} \frac{-i}{\frac{1}{2M}p^2 - P - i\epsilon}
\end{aligned} \tag{6.44}$$

which we recognize as the product of the four propagators and the vertex factor

$$\frac{-ie_0^2}{M} (2\pi)^5 g_{\mu\nu} \delta^4(k - k' - p' + p) \delta(\sigma\kappa - \sigma\kappa' + P' - P) \tag{6.45}$$

We may summarize the Feynman rules for the momentum space Green's functions as follows:

1. For each matter field propagator, draw a directed line associated with the factor

$$\frac{1}{(2\pi)^5} \frac{-i}{\frac{1}{2M}p^2 - P - i\epsilon}$$

2. For each photon propagator, draw a photon line associated with the factor

$$\frac{1}{\lambda} \mathcal{P}^{\mu\nu} \frac{-i}{k^2 + \sigma\kappa^2 - i\epsilon}$$

3. For the three-particle interaction, write the vertex factor

$$\frac{e_0}{2M} i(p + p')^\nu (2\pi)^5 \delta^4(p - p' - k) \delta(P - P' + \sigma\kappa)$$

4. For the four-particle interaction, write the vertex factor

$$\frac{-ie_0^2}{M} (2\pi)^5 g_{\mu\nu} \delta^4(k - k' - p' + p) \delta(\sigma\kappa - \sigma\kappa' + P' - P)$$

To obtain the Feynman rules for the S-matrix elements, we use the reduction formulas together with the free propagators for the incoming and outgoing particles. From (6.27) for the matter fields, we see that inserting a propagator will precisely cancel the derivative operator, so that in calculating S-matrix elements, the incoming and outgoing propagators are replaced by 1.

Using (6.31) for the photons, we see that the derivative operator will cancel the factor $-i/(k^2 + \sigma\kappa^2 - i\epsilon)$, but not the factor $\frac{1}{\lambda}$. So in the rules for the S-matrix elements we replace the incoming and outgoing photon propagators with the factor

$$\pm \frac{ig(s)}{(2\pi)^4\lambda} \varepsilon^\mu(k, s) \quad (6.46)$$

where the incoming photon takes the sign $-$ and the outgoing photon takes the sign $+$.

7 Scattering Cross-Sections

In this section, we calculate the relationship of the scattering cross-section to the transition amplitudes calculated from the S-matrix expansions discussed in the previous section. In particular, we specialize the five-dimensional quantum theory to the case of scattering. The development here generally follows the presentation given in [40], with elements taken from [42] and [43].

The initial state will be represented by constructing wave packets of the form

$$|\Psi_p\rangle = \int \frac{d^4q}{(2\pi)^4} \tilde{\psi}_p(q) |q\rangle = \int \frac{d^4q}{(2\pi)^4} \tilde{\psi}(q-p) |q\rangle \quad (7.1)$$

in which we take $\tilde{\psi}_p(q) = \tilde{\psi}(q-p)$ to be a narrow distribution centered around the momentum p . These wave packets correspond to linear superpositions of solutions to the quantum mechanical eigenvalue equation

$$K\psi_q = \kappa_q\psi_q \quad (7.2)$$

where

$$K = \frac{p^2}{2M} + V \quad \text{and} \quad \kappa_q = \frac{q^2}{2M} + \delta\kappa_q. \quad (7.3)$$

The wavefunctions may be written in the form

$$\Psi_p(x) = \int \frac{d^4q}{(2\pi)^4} \tilde{\psi}(q-p) e^{iq\cdot x} = e^{ip\cdot x} \int \frac{d^4\rho}{(2\pi)^4} \tilde{\psi}(\rho) e^{i\rho\cdot x} = e^{ip\cdot x} G(x) \quad (7.4)$$

in which $G(x)$ is a slowly varying, localized function of x . The τ -dependent wavefunctions are

$$\Psi(x, \tau) = e^{-iK\tau} \Psi_\rho(x) = \int \frac{d^4 q}{(2\pi)^4} \tilde{\psi}(q - p) e^{i(q \cdot x - \kappa_q \tau)}. \quad (7.5)$$

Since the momentum distribution is narrowly centered around p , the wavepacket in (7.5), may be expanded in $q = p + \rho$. To do this note that

$$\kappa_q = \kappa_p + \rho^\mu \partial_{p^\mu} \kappa_p + \frac{1}{2} \rho^\mu \rho^\nu \partial_{p^\mu} \partial_{p^\nu} \kappa_p + \cdots = \kappa_p + \rho \cdot u + o(\rho^2) \quad (7.6)$$

where $u^\sigma = \partial \kappa_p / \partial p_\sigma$ is the group 4-velocity. Then, to first order in ρ ,

$$\begin{aligned} \Psi(x, \tau) &= \int \frac{d^4 \rho}{(2\pi)^4} e^{i[(p+\rho) \cdot x - (\kappa_p + \rho \cdot u) \tau]} \tilde{\psi}(\rho) \\ &= e^{i[p \cdot x - \kappa_p \tau]} \int \frac{d^4 \rho}{(2\pi)^4} \tilde{\psi}(\rho) e^{i\rho \cdot [x - u\tau]} \\ &= e^{i[p \cdot x - \kappa_p \tau]} G(x - u\tau) \end{aligned} \quad (7.7)$$

and the slowly varying function $G(x - u\tau)$ describes the Ehrenfest motion of the wavepacket. The event density corresponding to the wavefunction in (7.7) is given by the integral of the current

$$j^4(x, \tau) = \bar{\Psi}_p \Psi_p = |G(x - u\tau)|^2. \quad (7.8)$$

In scattering, the initial state is a product of the target state and the beam state, and is represented by a wavepacket of the form

$$|i\rangle = |\Psi_{\text{target}}\rangle |\Psi_{\text{beam}}\rangle = \int \frac{d^4 q_T}{(2\pi)^4} \frac{d^4 q_B}{(2\pi)^4} \tilde{\psi}_T(q_T - p_T) \tilde{\psi}_B(q_B - p_B) |q_T q_B\rangle. \quad (7.9)$$

Writing the transition matrix T in terms of the scattering matrix S [15],

$$\begin{aligned} S &= 1 + iT \\ \langle f | S | i \rangle &= \delta_{if} + i \langle f | T | i \rangle \\ &= \delta_{if} + i(2\pi)^5 \delta(\kappa_f - \kappa_i) \delta^4(p_f - p_i) \langle f | T | i \rangle \end{aligned} \quad (7.10)$$

we find for the initial state defined in (7.9),

$$\langle f | T | i \rangle = \int \frac{d^4 q_T}{(2\pi)^4} \frac{d^4 q_B}{(2\pi)^4} \tilde{\psi}_T(q_T - p_T) \tilde{\psi}_B(q_B - p_B) \langle f | T | q_T q_B \rangle. \quad (7.11)$$

The transition probability becomes

$$|\langle f|\mathcal{T}|i\rangle|^2 = \int \frac{d^4 q_{\text{T}}}{(2\pi)^4} \frac{d^4 q'_{\text{T}}}{(2\pi)^4} \frac{d^4 q_{\text{B}}}{(2\pi)^4} \frac{d^4 q'_{\text{B}}}{(2\pi)^4} \tilde{\psi}_{\text{T}}(q_{\text{T}} - p_{\text{T}}) \tilde{\psi}_{\text{T}}^*(q'_{\text{T}} - p_{\text{T}}) \tilde{\psi}_{\text{B}}(q_{\text{B}} - p_{\text{B}}) \tilde{\psi}_{\text{B}}^*(q'_{\text{B}} - p_{\text{B}}) \\ \langle f|\mathcal{T}|q_{\text{T}}q_{\text{B}}\rangle \langle f|\mathcal{T}|q'_{\text{T}}q'_{\text{B}}\rangle^* \delta(\kappa_f - \kappa_i) \delta^4(p_f - p_i) \delta(\kappa_f - \kappa'_i) \delta^4(p_f - p'_i). \quad (7.12)$$

where

$$p_i = q_{\text{T}} + q_{\text{B}} = p_{\text{T}} + p_{\text{B}} + \rho_{\text{T}} + \rho_{\text{B}} \quad (7.13)$$

$$p'_i = q'_{\text{T}} + q'_{\text{B}} = p_{\text{T}} + p_{\text{B}} + \rho'_{\text{T}} + \rho'_{\text{B}} \quad (7.14)$$

$$\kappa_i = \kappa_{\text{T}} + \kappa_{\text{B}} = \frac{q_{\text{T}}^2}{2M_{\text{T}}} + \frac{q_{\text{B}}^2}{2M_{\text{B}}}. \quad (7.15)$$

We assume that the interaction occurs close to the central momenta, so that

$$\langle f|\mathcal{T}|q_{\text{T}}q_{\text{B}}\rangle \langle f|\mathcal{T}|q'_{\text{T}}q'_{\text{B}}\rangle^* \simeq |\langle f|\mathcal{T}|p_{\text{T}}p_{\text{B}}\rangle|^2 = |\mathcal{T}_{fi}|^2. \quad (7.16)$$

We may re-write the δ -functions, replacing $q = p + \rho$ for each momentum:

$$\begin{aligned} \delta^4(p_f - p_i) \delta^4(p_f - p'_i) &= \delta^4(p_f - p_{\text{T}} - \rho_{\text{T}} - p_{\text{B}} - \rho_{\text{B}}) \\ &\quad \delta^4(p_f - p_{\text{T}} - \rho'_{\text{T}} - p_{\text{B}} - \rho'_{\text{B}}) \\ &= \delta^4(p_f - p_{\text{T}} - p_{\text{B}} - (\rho_{\text{T}} + \rho_{\text{B}})) \\ &\quad \delta^4(p_f - p_{\text{T}} - p_{\text{B}} - (\rho'_{\text{T}} + \rho'_{\text{B}})) \\ &= \delta^4(p_f - p_{\text{T}} - p_{\text{B}} - (\rho_{\text{T}} + \rho_{\text{B}})) \\ &\quad \delta^4(\rho_{\text{T}} + \rho_{\text{B}} - \rho'_{\text{T}} - \rho'_{\text{B}}) \\ &\simeq \delta^4(p_f - p_{\text{T}} - p_{\text{B}}) \\ &\quad \delta^4(\rho_{\text{T}} - \rho'_{\text{T}} + \rho_{\text{B}} - \rho'_{\text{B}}). \end{aligned} \quad (7.17)$$

Similarly,

$$\delta(\kappa_f - \kappa_i) \delta(\kappa_f - \kappa'_i) \simeq \delta(\kappa_{p_f} - \kappa_{p_{\text{T}}} - \kappa_{p_{\text{B}}}) \delta(\kappa_{\rho_{\text{T}}} + \kappa_{\rho_{\text{B}}} - \kappa_{\rho'_{\text{T}}} - \kappa_{\rho'_{\text{B}}}). \quad (7.18)$$

so that, since $d^4 q = d^4 \rho$, (7.12) becomes

$$|\langle f|\mathcal{T}|i\rangle|^2 = (2\pi)^{10} \delta(\kappa_{p_f} - \kappa_{p_{\text{T}}} - \kappa_{p_{\text{B}}}) \delta^4(p_f - p_{\text{T}} - p_{\text{B}}) |\mathcal{T}_{fi}|^2 \\ \int \frac{d^4 \rho_{\text{T}}}{(2\pi)^4} \frac{d^4 \rho'_{\text{T}}}{(2\pi)^4} \frac{d^4 \rho_{\text{B}}}{(2\pi)^4} \frac{d^4 \rho'_{\text{B}}}{(2\pi)^4} \tilde{\psi}_{\text{T}}(\rho_{\text{T}}) \tilde{\psi}_{\text{T}}^*(\rho'_{\text{T}}) \tilde{\psi}_{\text{B}}(\rho_{\text{B}}) \tilde{\psi}_{\text{B}}^*(\rho'_{\text{B}}) \delta^4(\rho_{\text{T}} - \rho'_{\text{T}} + \rho_{\text{B}} - \rho'_{\text{B}}) \\ \delta(\kappa_{\rho_{\text{T}}} + \kappa_{\rho_{\text{B}}} - \kappa_{\rho'_{\text{T}}} - \kappa_{\rho'_{\text{B}}}). \quad (7.19)$$

Notice that since $\kappa_\rho \simeq \rho \cdot u$,

$$\begin{aligned}
\kappa_{\rho_T} - \kappa_{\rho'_T} + \kappa_{\rho_B} - \kappa_{\rho'_B} &\simeq \rho_T \cdot u_T - \rho'_T \cdot u_T + \rho_B \cdot u_B - \rho'_B \cdot u_B \\
&= (\rho_T - \rho'_T) \cdot u_T + (\rho_B - \rho'_B) \cdot u_B \\
&= (u_B - u_T) \cdot (\rho_B - \rho'_B) + u_T \cdot (\rho_B - \rho'_B + \rho_T - \rho'_T) \\
&= (u_B - u_T) \cdot (\rho_B - \rho'_B)
\end{aligned} \tag{7.20}$$

where we have used (7.17) in the last line of (7.20). To continue, we write integral representations for the δ -functions. Thus,

$$\delta^4(\rho_T - \rho'_T + \rho_B - \rho'_B) = \frac{1}{(2\pi)^4} \int d^4x e^{ix \cdot (\rho_T - \rho'_T + \rho_B - \rho'_B)} \tag{7.21}$$

and

$$\delta((u_B - u_T) \cdot (\rho_B - \rho'_B)) = \int \frac{d\alpha}{2\pi} e^{i\alpha(u_B - u_T) \cdot (\rho_B - \rho'_B)}. \tag{7.22}$$

Inserting (7.20), (7.21) and (7.22) into (7.19), we have

$$\begin{aligned}
|\langle f | T | i \rangle|^2 &= (2\pi)^{10} \delta(\kappa_{p_f} - \kappa_{p_T} - \kappa_{p_B}) \delta^4(p_f - p_T - p_B) |\mathcal{T}_{fi}|^2 \\
&\quad \frac{1}{(2\pi)^5} \int d^4x d\alpha \frac{d^4\rho_T}{(2\pi)^4} \frac{d^4\rho'_T}{(2\pi)^4} \frac{d^4\rho_B}{(2\pi)^4} \frac{d^4\rho'_B}{(2\pi)^4} \tilde{\psi}_T(\rho_T) \tilde{\psi}_T^*(\rho'_T) \tilde{\psi}_B(\rho_B) \tilde{\psi}_B^*(\rho'_B) \\
&\quad e^{ix \cdot (\rho_T - \rho'_T + \rho_B - \rho'_B)} e^{i\alpha(u_B - u_T) \cdot (\rho_B - \rho'_B)} \\
&= (2\pi)^5 \delta(\kappa_{p_f} - \kappa_{p_T} - \kappa_{p_B}) \delta^4(p_f - p_T - p_B) |\mathcal{T}_{fi}|^2 \\
&\quad \int d^4x d\alpha |G_T(x)|^2 \left| G_B(x + \alpha(u_B - u_T)) \right|^2.
\end{aligned} \tag{7.23}$$

The transition probability per unit spacetime volume is then

$$\begin{aligned}
\frac{d}{dV dT} |\langle f | T | i \rangle|^2 &= (2\pi)^5 \delta(\kappa_{p_f} - \kappa_{p_T} - \kappa_{p_B}) \delta^4(p_f - p_T - p_B) |\mathcal{T}_{fi}|^2 \\
&\quad \int d\alpha |G_T(x)|^2 \left| G_B(x + \alpha(u_B - u_T)) \right|^2 \\
&= (2\pi)^5 \delta(\kappa_{p_f} - \kappa_{p_T} - \kappa_{p_B}) \delta^4(p_f - p_T - p_B) |\mathcal{T}_{fi}|^2 |G_T(x)|^2 \\
&\quad \int d\alpha \left| G_B(\alpha(u_B - u_T)) \right|^2.
\end{aligned} \tag{7.24}$$

We take the target-beam axis to be along the z -axis. Then, the relative group 4-velocity becomes

$$u_B - u_T = (u_B^0 - u_T^0, 0, 0, u_B^3 - u_T^3) \tag{7.25}$$

and so

$$\int d\alpha \left| G_B(\alpha(u_B - u_T)) \right|^2 = \int d\alpha \left| G_B(\alpha(u_B^0 - u_T^0), 0, 0, \alpha(u_B^3 - u_T^3)) \right|^2. \tag{7.26}$$

Making the change of variable $\xi = \alpha(u_B^3 - u_T^3)$, this becomes

$$\int d\alpha \left| G_B(\alpha(u_B - u_T)) \right|^2 = \frac{1}{(u_B^3 - u_T^3)} \int d\xi \left| G_B\left(\frac{\xi}{v}, 0, 0, \xi\right) \right|^2 \quad (7.27)$$

where

$$v = \frac{u_B^3 - u_T^3}{u_B^0 - u_T^0} \quad (7.28)$$

corresponds to the relative speed ($\sim dx/dt$) of the target and beam events. By (7.8), and since $G(x - u\tau)$ satisfies

$$\partial_\tau G(x - u\tau) = -u^\mu \partial_{x^\mu} G(x - u\tau). \quad (7.29)$$

we may take j^μ to be

$$j^\mu = u^\mu G(x - u\tau) \quad (7.30)$$

In order to understand the integral over ξ in (7.27), we notice that for $u^3 = u_B^3 - u_T^3 > 0$, the relative motion will cross from $x^3 < 0$ to $x^3 > 0$ during the scattering. We define by $N^+(\tau)$ the proportion of all events for which $x^3 > 0$, given by

$$N^+(\tau) = \int_0^\infty dx^3 \int dt d^2 x_\perp j^4(x, \tau) \quad (7.31)$$

where x_\perp refers to the 1-2 plane. The rate at which events cross $x^3 = 0$ is then given by

$$\begin{aligned} \frac{d}{d\tau} N^+(\tau) &= \int_0^\infty dx^3 \int dt d^2 x_\perp \partial_\tau j^4(x, \tau) \\ &= \int_0^\infty dx^3 \int_{-\infty}^\infty dt \int d^2 x_\perp [-u^\mu \partial_\mu j^4(x, \tau)] \\ &= \int_0^\infty dx^3 \int dt d^2 x_\perp [-u^3 \partial_3 j^4(x, \tau)] \end{aligned} \quad (7.32)$$

where in the last line, we used $j^4(x, \tau) \rightarrow 0$ as $x^\mu \rightarrow \pm\infty$. Evaluating the integral on dx^3 , we find

$$\frac{d}{d\tau} N^+(\tau) = -u^3 \int dt d^2 x_\perp j^4(t, x_\perp, x^3, \tau) \Big|_{x^3=0}^{x^3=\infty} = u^3 \int dt d^2 x_\perp j^4(t, x_\perp, 0, \tau). \quad (7.33)$$

In terms of (7.32) and (7.8), the total number of events which cross $x^3 = 0$ for all τ is given by

$$\begin{aligned} N^+(\infty) - N^+(-\infty) &= \int d\tau \frac{d}{d\tau} N^+(\tau) \\ &= \int d\tau u^3 \int dt d^2 x_\perp j^4(t, x_\perp, 0, \tau) \\ &= \int d\tau u^3 \int dt d^2 x_\perp \left| G\left((t, x_\perp, 0) - (u^0, 0, 0, u^3)\tau\right) \right|^2 \\ &= \int d\tau u^3 \int dt d^2 x_\perp \left| G\left(t - u^0\tau, x_\perp, -u^3\tau\right) \right|^2. \end{aligned} \quad (7.34)$$

Making the change of variables $\xi = u^3\tau$ puts (7.34) into the form

$$\begin{aligned} N^+(\infty) - N^+(-\infty) &= \int d\xi \int dt d^2x_\perp \left| G\left(t - \frac{u^0}{u^3}\xi, x_\perp, -\xi\right) \right|^2 \\ &= \int d\xi \int dt d^2x_\perp \left| G\left(t + \frac{1}{v}\xi, x_\perp, \xi\right) \right|^2. \end{aligned} \quad (7.35)$$

in which v is given by (7.28). The total number of events which cross $x^3 = 0$, per unit area (in the 1-2 plane) per unit time (which defines the 3-flux $F^{(3)}$ of the beam) is just,

$$F^{(3)} = \int d\xi \left| G\left(\frac{1}{v}\xi, 0, 0, \xi\right) \right|^2. \quad (7.36)$$

Comparison with (7.27) shows that

$$\int d\alpha \left| G_B\left(\alpha(u_B - u_T)\right) \right|^2 = \frac{1}{|\vec{u}_B - \vec{u}_T|} F^{(3)} \quad (7.37)$$

Therefore, we may re-write (7.24) as

$$\begin{aligned} \frac{d}{dV dT} |\langle f | T | i \rangle|^2 &= (2\pi)^5 \delta(\kappa_{p_f} - \kappa_{p_T} - \kappa_{p_B}) \delta^4(p_f - p_T - p_B) |\mathcal{T}_{fi}|^2 |G_T(x)|^2 \\ &\quad \frac{1}{|\vec{u}_B - \vec{u}_T|} F^{(3)}. \end{aligned} \quad (7.38)$$

The scattering 3-cross-section is defined through

$$\frac{d}{dV dT} |\langle f | T | i \rangle|^2 = d\sigma^{(3)} \times \text{beam flux} \times \text{target density} \quad (7.39)$$

and so, we find that

$$d\sigma^{(3)} = (2\pi)^5 \delta(\kappa_{p_f} - \kappa_{p_T} - \kappa_{p_B}) \delta^4(p_f - p_T - p_B) \frac{|\langle f | \mathcal{T} | p_T p_B \rangle|^2}{|\vec{u}_B - \vec{u}_T|}. \quad (7.40)$$

The Process $B + T \rightarrow 1 + 2$

We will find it convenient to treat the case of two particle final states in relative coordinates, and we make the following definitions

$$P = \frac{1}{2}(p_T + p_B) \quad p = p_T - p_B \quad (7.41)$$

$$P' = \frac{1}{2}(p_1 + p_2) \quad p' = p_1 - p_2 \quad (7.42)$$

which have the inverse relations

$$p_T = P + \frac{1}{2}p \quad p_B = P - \frac{1}{2}p \quad (7.43)$$

$$p_1 = P' + \frac{1}{2}p' \quad p_2 = P' - \frac{1}{2}p' \quad (7.44)$$

In the center of mass system, we have that

$$\vec{p}_B + \vec{p}_T = \vec{p}_1 + \vec{p}_2 = 0 \quad (7.45)$$

so that

$$P = \frac{1}{2} \begin{bmatrix} E(\vec{p}_B) + E(\vec{p}_T) \\ \vec{p}_B + \vec{p}_T \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[E(\vec{p}_B) + E(\vec{p}_T)] \\ \vec{0} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{s} \\ \vec{0} \end{bmatrix} \quad (7.46)$$

where

$$s = -(p_B + p_T)^2 \quad (7.47)$$

is the usual Mandelstam parameter in this metric (and so $P^2 = -s/4$), and

$$E(\vec{p}) = \sqrt{(\vec{p})^2 + m^2}. \quad (7.48)$$

Similarly,

$$p' = p_1 - p_2 = \begin{bmatrix} E(p_1) - E(p_2) \\ 2\vec{p}_1 \end{bmatrix} \quad (7.49)$$

and

$$\begin{aligned} p_1^2 &= P^2 + \frac{1}{4}(p')^2 + P \cdot p' = -\frac{1}{4}s + \frac{1}{4}(p')^2 - \frac{1}{2}\sqrt{s} [E(p_1) - E(p_2)] \\ p_2^2 &= P^2 + \frac{1}{4}(p')^2 - P \cdot p' = -\frac{1}{4}s + \frac{1}{4}(p')^2 + \frac{1}{2}\sqrt{s} [E(p_1) - E(p_2)]. \end{aligned} \quad (7.50)$$

Since the relative momentum is spacelike, we may introduce the parameterization

$$p' = \rho \begin{bmatrix} \sinh \beta \\ \cosh \beta \hat{n} \end{bmatrix} \quad (7.51)$$

so that (7.50) becomes

$$p_1^2 = -\frac{1}{4}(s - \rho^2 + 2\sqrt{s}\rho \sinh \beta) \quad p_2^2 = -\frac{1}{4}(s - \rho^2 - 2\sqrt{s}\rho \sinh \beta). \quad (7.52)$$

We may see the utility of this approach in the conventional description of relativistic scattering, where the cross-section has the $O(3,1)$ invariant measure

$$dR_2 = \delta^4(p_1 + p_2 - p_T - p_B) \delta(p_1^2 + m_1^2) \delta(p_2^2 + m_2^2) \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \quad (7.53)$$

which in relative coordinates becomes

$$\begin{aligned} dR_2 &= \frac{1}{2(2\pi)^8} \delta^4(P - P') \delta(p_1^2 + m_1^2) \delta(p_2^2 + m_2^2) d^4 P' d^4 p' \\ &= \frac{1}{2(2\pi)^8} \delta(p_1^2 + m_1^2) \delta(p_2^2 + m_2^2) d^4 p' \\ &= \frac{1}{2(2\pi)^8} \cdot 4 \cdot 4 \cdot \delta(s - \rho^2 + 2\sqrt{s}\rho \sinh \beta - 4m_1^2) \delta(s - \rho^2 - 2\sqrt{s}\rho \sinh \beta - 4m_2^2) \\ &\quad \rho^3 \cosh^2 \beta d\rho d\beta d\Omega \\ &= \frac{4}{(2\pi)^8} \rho^3 \cosh^2 \beta d\rho d\beta d\Omega \delta(\rho^2 - s + (m_1^2 + m_2^2)) \delta(s - \rho^2 - 2\sqrt{s}\rho \sinh \beta - 4m_2^2) \\ &= \frac{\rho}{(2\pi)^8 \sqrt{s}} \cosh^2 \beta d\rho d\beta d\Omega \delta\left[\rho - \sqrt{s - 2(m_1^2 + m_2^2)}\right] \delta\left[\sinh \beta - \frac{m_1^2 - m_2^2}{\sqrt{s}\rho}\right] \end{aligned} \quad (7.54)$$

Changing variables as

$$\rho \cosh^2 \beta d\beta = \rho \cosh \beta d(\sinh \beta) \quad (7.55)$$

and recognizing $\rho \cosh \beta$ as $|\vec{p}'| = 2|\vec{p}_f|$, this becomes,

$$dR_2 = \frac{2|\vec{p}_f|}{\sqrt{s}} d\Omega \quad (7.56)$$

with

$$\rho = \sqrt{s - 2(m_1^2 + m_2^2)} \quad (7.57)$$

so that

$$|\vec{p}_f| = \frac{\rho}{2} \sqrt{1 + \left[\frac{m_1^2 - m_2^2}{\sqrt{s}\rho}\right]^2} = \frac{1}{2\sqrt{s}} \sqrt{s^2 - 2s(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2} = \frac{\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)}{2\sqrt{s}} \quad (7.58)$$

in agreement with the usual derivation [40].

Returning to the off-shell theory, we first consider scattering of indistinguishable particles, which means that we may take the invariant mass parameter M to be the same for all events.

The invariant measure for the matter field is given by

$$\begin{aligned}
dR_2 &= \delta^4(p_1 + p_2 - p_T - p_B) \delta \left(\frac{(p_1)^2}{2M} + \frac{(p_2)^2}{2M} - \frac{(p_B)^2}{2M} - \frac{(p_T)^2}{2M} \right) \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \\
&= \frac{1}{2(2\pi)^8} \delta^4(P - P') d^4 P' \ 2M \delta \left((p_1)^2 + (p_2)^2 - (p_B)^2 - (p_T)^2 \right) d^4 p' \\
&= \frac{2M}{(2\pi)^8} \delta(p^2 - p'^2) d^4 p'
\end{aligned} \tag{7.59}$$

where we have used

$$\begin{aligned}
(p_1)^2 + (p_2)^2 &= P'^2 + \frac{1}{4} p'^2 + P' \cdot p' + P'^2 + \frac{1}{4} p'^2 - P' \cdot p' \\
&= 2P'^2 + \frac{1}{2} p'^2
\end{aligned} \tag{7.60}$$

$$(p_B)^2 + (p_T)^2 = 2P^2 + \frac{1}{2} p^2 \tag{7.61}$$

Now using the parameterization in (7.51), we find

$$\begin{aligned}
dR_2 &= \frac{2M}{(2\pi)^8} \delta(p^2 - \rho^2) \rho^3 \cosh^2 \beta d\rho d\beta d\Omega \\
&= \frac{2M}{(2\pi)^8} \frac{1}{2\rho} \delta \left(\sqrt{p^2} - \rho \right) \rho^3 \cosh^2 \beta d\rho d\beta d\Omega \\
&= \frac{M}{(2\pi)^8} \delta \left(\sqrt{p^2} - \rho \right) (\rho \cosh \beta)^2 d\rho d\beta d\Omega \\
&= \frac{M}{(2\pi)^8} \delta \left(\sqrt{p^2} - \rho \right) (2|\vec{p}_f|)^2 d\rho d\beta d\Omega
\end{aligned} \tag{7.62}$$

where the absolute value of

$$\vec{p}_f = \vec{p}_1 = -\vec{p}_2 = \frac{1}{2} \rho \cosh \beta \hat{n} \tag{7.63}$$

is undetermined in the off-shell theory, because the value of β is not fixed by conservation of mass-momentum. But notice that ρ is determined by

$$\begin{aligned}
\rho^2 &= p^2 = (p_B - p_T)^2 = (p_B)^2 + (p_T)^2 - 2p_B \cdot p_T \\
&= -(p_B)^2 - (p_T)^2 - 2p_B \cdot p_T + 2[(p_B)^2 + (p_T)^2] \\
&= -(p_B + p_T)^2 + 2[(p_B)^2 + (p_T)^2] \\
&= s - 2[(m_B)^2 + (m_T)^2]
\end{aligned} \tag{7.64}$$

which is identical to the usual on-shell value given in (7.57). Combining (7.62) with (7.40), we have for the scattering cross-section (see also [15])

$$d\sigma^{(3)} = \frac{(2\pi)^5}{(2\pi)^8} \frac{|\langle p_1 p_2 | \mathcal{T} | p_T p_B \rangle|^2}{|\vec{u}_B - \vec{u}_T|} 4M |\vec{p}_f|^2 d\beta d\Omega. \tag{7.65}$$

Since $p = Mu$ for the asymptotic free matter field, we may write this in the form

$$d\sigma^{(3)} = \frac{1}{(2\pi)^3} M^2 \frac{|\vec{p}_f|^2}{|\vec{p}_i|} |\langle p_1 p_2 | \mathcal{T} | p_T p_B \rangle|^2 d\beta d\Omega. \quad (7.66)$$

where $\vec{p}_i = \vec{p}_T - \vec{p}_B$, and we must keep in mind that \vec{p}_f is dependent on β .

We denote by $\Delta m^2 = (m_1)^2 - (m_2)^2$, and notice that from (7.52),

$$\Delta m^2 = (p_2)^2 - (p_1)^2 = \sqrt{s} \rho \sinh \beta \quad (7.67)$$

so that

$$d(\Delta m^2) = \sqrt{s} \rho \cosh \beta d\beta = \sqrt{s} 2 |\vec{p}_f| d\beta. \quad (7.68)$$

Therefore, the cross-section can be written as

$$d\sigma^{(3)} = \frac{1}{(2\pi)^3} M^2 \frac{|\vec{p}_f|}{2\sqrt{s} |\vec{p}_i|} |\langle p_1 p_2 | \mathcal{T} | p_T p_B \rangle|^2 d\Omega d(\Delta m^2). \quad (7.69)$$

We may compare the cross-section with the comparable expression in the usual on-shell theory, which can be written in the center of mass as [40]

$$\frac{d\sigma^{(2)}}{d\Omega} = \frac{1}{64\pi^2} \frac{|\vec{p}_f|}{s |\vec{p}_i|} |\langle p_1 p_2 | \mathcal{T} | p_T p_B \rangle|^2. \quad (7.70)$$

Putting (7.69) into a form similar to (7.70), we have

$$\frac{d\sigma^{(3)}}{d\Omega} = \frac{1}{64\pi^2} \frac{|\vec{p}_f|}{s |\vec{p}_i|} |\langle p_1 p_2 | \mathcal{T} | p_T p_B \rangle|^2 d\left(\frac{4M^2\sqrt{s}}{\pi} \Delta m^2\right). \quad (7.71)$$

Examining the Feynman rules for the single photon interaction in Section 6, we see that the usual vertex factor of e is replaced by $e_0/2M = \lambda e/2M$ and photon propagator has an extra factor of $1/\lambda$, so that the squared transition amplitude will have a factor of $\lambda^2/16M^4$ in relation to the usual on-shell case. Combining this factor with the extra factor in (7.71), we have an overall extra factor of

$$\frac{\lambda^2}{16M^4} d\left(\frac{4M^2\sqrt{s}}{\pi} \Delta m^2\right) = d\left(\frac{\sqrt{s}\lambda^2}{4\pi M^2} \Delta m^2\right) \quad (7.72)$$

which we see has the expected units of length. We may understand this factor in the following way: During the scattering, the event will propagate a distance $\Delta x \sim (p/M)\Delta\tau$, so that the uncertainty relation tells us that

$$\Delta\tau \sim \frac{\Delta x}{(p/M)} = \frac{M}{p} \Delta x \sim \frac{1}{2\Delta p} \frac{M}{p} \quad (7.73)$$

So from $(\vec{p})^2 = E^2 - m^2$, we find that

$$\vec{p} \cdot d\vec{p} \sim m \, dm \sim \frac{1}{2} \Delta(m^2) \quad (7.74)$$

and combining with (7.73),

$$\Delta\tau \sim \frac{M}{\Delta(m^2)}. \quad (7.75)$$

Shnerb and Horwitz [25] have shown that for $\Delta\tau > \lambda$, the photon-current interaction becomes uncorrelated, so we may take $\lambda \sim \Delta\tau$. Thus, the factor in (7.72) becomes

$$\begin{aligned} d\left(\frac{\sqrt{s}\lambda^2}{4\pi M^2} \Delta(m^2)\right) &\sim d\left(\frac{\sqrt{s}(\Delta\tau)^2}{4\pi M} \frac{\Delta(m^2)}{M}\right) \sim d\left(\frac{\sqrt{s}(\Delta\tau)^2}{4\pi M} \frac{1}{\Delta\tau}\right) = d\left(\frac{\sqrt{s}(\Delta\tau)}{4\pi M}\right) \\ &\sim d\left(\Delta\tau \frac{E}{4\pi M}\right) \sim \frac{1}{4\pi} \frac{dt}{d\tau} d\tau \sim dt_{\text{shift}}, \end{aligned} \quad (7.76)$$

where we understand dt_{shift} as the change in the relative time coordinate of the two particles during the scattering. Thus, the usual on-shell scattering cross-section corresponds to a specific dt_{int} , and we may compare the off-shell cross-sections with the usual results through the expression.

$$\frac{d\sigma^{(3)}}{d\Omega \, dt_{\text{int}}} = \frac{1}{64\pi^2} \frac{|\vec{p}_f|}{s |\vec{p}_i|} |\langle p_1 p_2 | \tilde{T} | p_T p_B \rangle|^2, \quad (7.77)$$

where $\tilde{T} = \left(\frac{\lambda}{4M^2}\right)^2$.

8 Scattering Processes

In this section, we will carry out explicit calculations, in two ways, of the transition amplitude for specific scattering processes, and show how the results differ from the usual treatment in on-shell quantum field theory. The first calculation is made explicitly from the S-matrix expansion (6.17), and subsequent calculations are made on the basis of the Feynman rules.

Compton Scattering

We begin with the interaction Hamiltonian (6.35) and we take the initial and final states to each consist of one electron and one photon. Thus,

$$|\text{in}\rangle = |p \, k, s\rangle = b^*(p) \, a^*(k, s)|0\rangle \quad |\text{out}\rangle = |p' \, k', s'\rangle = b^*(p') \, a^*(k', s')|0\rangle \quad (8.1)$$

To lowest order (tree level), we have

$$\begin{aligned}
\langle \text{out} | S | \text{in} \rangle &= \langle \text{out} | T e^{-i \int d^5 y \mathcal{K}_I} | \text{in} \rangle \\
&= \langle \text{out} | i \int d^5 y \frac{e_0^2}{2M} a_\mu(y) a^\mu(y) \psi(y) \psi^*(y) | \text{in} \rangle + \\
&\quad \langle \text{out} | \frac{1}{2!} \int d^5 y d^5 y' \left[\frac{i e_0}{2M} \right]^2 T a_\mu(y) j^\mu(y) a_\nu(y') j^\nu(y') | \text{in} \rangle
\end{aligned} \tag{8.2}$$

where j^μ is given in (6.37) and $d^5 y = d^4 y d\tau$. The τ -ordered product in the second term gives

$$\begin{aligned}
\int d^5 y d^5 y' T a_\mu(y) j^\mu(y) a_\nu(y') j^\nu(y') &= \int d^5 y d^5 y' [\theta(\tau - \tau') a_\mu(y) a_\nu(y') j^\mu(y) j^\nu(y') + \\
&\quad \theta(\tau' - \tau) a_\nu(y') a_\mu(y) j^\nu(y') j^\mu(y)]
\end{aligned} \tag{8.3}$$

and

$$\begin{aligned}
\langle \text{out} | \int d^5 y d^5 y' \theta(\tau - \tau') a_\mu(y) a_\nu(y') j^\mu(y) j^\nu(y') | \text{in} \rangle &= \\
&= \int d^5 y d^5 y' \theta(\tau - \tau') \langle 0 | a(k' s') b(p') a_\mu(y) a_\nu(y') j^\mu(y) j^\nu(y') b^*(p) a^*(k, s) | 0 \rangle \\
&= \int d^5 y d^5 y' \theta(\tau - \tau') \langle 0 | a(k' s') a_\mu(y) a_\nu(y') a^*(k, s) | 0 \rangle \langle 0 | b(p') j^\mu(y) j^\nu(y') b^*(p) | 0 \rangle .
\end{aligned} \tag{8.4}$$

Using (5.14) for $a_\mu(y)$ and keeping in mind that only terms with equal numbers of creation and annihilation operators will contribute to vacuum expectation values, we have

$$\begin{aligned}
\langle 0 | a(k' s') a_\mu(y) a_\nu(y') a^*(k, s) | 0 \rangle &= \\
&\sum_{r, r'} \int \frac{d^4 q}{2\kappa_q} \frac{d^4 q'}{2\kappa_{q'}} \varepsilon_r^\mu(q) \varepsilon_{r'}^\nu(q') [\langle 0 | a(k' s') a^*(q', r') a(q, r) a^*(k, s) | 0 \rangle e^{i(q \cdot y + \sigma \kappa_q \tau)} e^{-i(q' \cdot y' + \sigma \kappa_{q'} \tau')} + \\
&\quad \langle 0 | a(k' s') a^*(q, r) a(q', r') a^*(k, s) | 0 \rangle e^{i(q' \cdot y' + \sigma \kappa_{q'} \tau')} e^{-i(q \cdot y + \sigma \kappa_q \tau)}].
\end{aligned} \tag{8.5}$$

Using the commutation relations (5.43) and discarding the disconnected terms (contracting k with k'), we get

$$\langle 0 | a(k' s') a^*(q, r) a(q', r') a^*(k, s) | 0 \rangle = \frac{2\kappa' g(s')}{(2\pi)^4 \lambda} \delta_{rs'} \delta^4(q - k') \frac{2\kappa g(s)}{(2\pi)^4 \lambda} \delta_{rs} \delta^4(q' - k) \tag{8.6}$$

so that (8.5) becomes

$$\begin{aligned}
\langle 0 | a(k' s') a_\mu(y) a_\nu(y') a^*(k, s) | 0 \rangle &= \sum_{r, r'} \int \frac{d^4 q}{2\kappa_q} \frac{d^4 q'}{2\kappa_{q'}} \varepsilon_r^\mu(q) \varepsilon_{r'}^\nu(q') \\
&\quad \left[\frac{2\kappa' g(s')}{(2\pi)^4 \lambda} \delta_{r's'} \delta^4(q' - k') \frac{2\kappa g(s)}{(2\pi)^4 \lambda} \delta_{rs} \delta^4(q - k) e^{i(q \cdot y + \sigma \kappa_q \tau)} e^{-i(q' \cdot y' + \sigma \kappa_{q'} \tau)} + \right.
\end{aligned}$$

$$\begin{aligned}
& \frac{2\kappa'g(s')}{(2\pi)^4\lambda}\delta_{rs'}\delta^4(q-k')\frac{2\kappa g(s)}{(2\pi)^4\lambda}\delta_{r's}\delta^4(q'-k)e^{i(q'\cdot y+\sigma\kappa_{q'}\tau)}e^{-i(q\cdot y+\sigma\kappa_q\tau)}] \\
&= \frac{g(s)g(s')}{(2\pi)^8\lambda^2}[\varepsilon_s^\mu(k)\varepsilon_{s'}^\nu(k')e^{i(k\cdot y+\sigma\kappa\tau)}e^{-i(k'\cdot y'+\sigma\kappa'\tau')} + \\
& \quad \varepsilon_{s'}^\mu(k')\varepsilon_s^\nu(k)e^{i(k\cdot y'+\sigma\kappa'\tau')}e^{-i(k'\cdot y+\sigma\kappa\tau)}]. \tag{8.7}
\end{aligned}$$

Using (4.6) and (6.37), the second vacuum expectation value is given by

$$\begin{aligned}
& \langle 0|b(p')j^\mu(y)j^\nu(y')b^*(p)|0\rangle = \\
&= \langle 0|b(p')[(\psi^*(y)\partial^\mu\psi(y) - (\partial^\mu\psi^*(y))\psi(y)] \\
& \quad [(\psi^*(y')\partial^\mu\psi(y') - (\partial^\mu\psi^*(y'))\psi(y'))]b^*(p)|0\rangle \\
&= -\frac{1}{(2\pi)^{16}}\int d^4qd^4q'd^4ld^4l'e^{-i(q\cdot y-\kappa_q\tau)}e^{i(q'\cdot y-\kappa_{q'}\tau)}e^{-i(l\cdot y'-\kappa_l\tau')}e^{i(l'\cdot y'-\kappa_{l'}\tau')} \\
& \quad (q+q')^\mu(l+l')^\nu\langle 0|b(p')b^*(q)b(q')b^*(l)b(l')b^*(p)|0\rangle. \tag{8.8}
\end{aligned}$$

Using the commutation relations (4.7), we find

$$\begin{aligned}
& \langle 0|b(p')j^\mu(y)j^\nu(y')b^*(p)|0\rangle = \\
&= -\int \frac{d^4q}{(2\pi)^4}(p+q)^\nu(p'+q)^\mu e^{-i(p'\cdot y-\kappa_{p'}\tau')}e^{i(q\cdot y-\kappa_q\tau)}e^{-i(q\cdot y'-\kappa_{q'}\tau')}e^{i(p\cdot y'-\kappa_p\tau')} \tag{8.9}
\end{aligned}$$

and similarly

$$\begin{aligned}
& \langle 0|b(p')j^\nu(y')j^\mu(y)b^*(p)|0\rangle = \\
&= -\int \frac{d^4q}{(2\pi)^4}(p+q)^\mu(p'+q)^\nu e^{-i(p'\cdot y'-\kappa_{p'}\tau')}e^{i(q\cdot y'-\kappa_{q'}\tau')}e^{-i(q\cdot y-\kappa_q\tau)}e^{i(p\cdot y-\kappa_p\tau)}. \tag{8.10}
\end{aligned}$$

Putting these terms together, we find

$$\begin{aligned}
& \langle \text{out}|\frac{1}{2!}\int d^5yd^5y'\left[\frac{ie_0}{2M}\right]^2\text{Ta}_\mu(y)j^\mu(y)a_\nu(y')j^\nu(y')|\text{in}\rangle = \\
&= -\frac{1}{2}\frac{g(s)g(s')}{(2\pi)^8\lambda^2}\left[\frac{e_0}{2M}\right]^2\int d^5yd^5y'\frac{d^4q}{(2\pi)^4} \\
& \quad [\varepsilon_s^\mu(k)\varepsilon_{s'}^\nu(k')e^{i(k\cdot y+\sigma\kappa\tau)}e^{-i(k'\cdot y'+\sigma\kappa'\tau')} + \varepsilon_{s'}^\mu(k')\varepsilon_s^\nu(k)e^{i(k\cdot y'+\sigma\kappa'\tau')}e^{-i(k'\cdot y+\sigma\kappa\tau)}] \\
& \quad [\theta(\tau-\tau')(p+q)^\nu(p'+q)^\mu e^{-i(p'\cdot y-\kappa_{p'}\tau)}e^{i(q\cdot y-\kappa_q\tau)}e^{-i(q\cdot y'-\kappa_{q'}\tau')}e^{i(p\cdot y'-\kappa_p\tau')} \\
& \quad + \theta(\tau'-\tau)(p+q)^\mu(p'+q)^\nu e^{-i(p'\cdot y'-\kappa_{p'}\tau')}e^{i(q\cdot y'-\kappa_{q'}\tau')}e^{-i(q\cdot y-\kappa_q\tau)}e^{i(p\cdot y-\kappa_p\tau)}]. \tag{8.11}
\end{aligned}$$

Performing the integrations on d^4y and d^4y' leads to δ -functions of p, p', k, k' and q , so that the integration on d^4q may be performed to arrive at

$$\langle \text{out}|\frac{1}{2!}\int d^5yd^5y'\left[\frac{ie_0}{2M}\right]^2\text{Ta}_\mu(y)j^\mu(y)a_\nu(y')j^\nu(y')|\text{in}\rangle =$$

$$\begin{aligned}
&= -\frac{1}{2} \frac{g(s)g(s')}{(2\pi)^4 \lambda^2} \left[\frac{e_0}{2M} \right]^2 \delta^4(p+k-p'-k') \int d\tau d\tau' \\
&\quad \{ \theta(\tau - \tau') [\varepsilon_s^\mu(k)(2p' - k)_\mu \varepsilon_{s'}^\nu(k')(2p - k')_\nu \\
&\quad \quad e^{i[\sigma\kappa + \kappa_{p'} - (p-k')^2/2M]\tau - i[\sigma\kappa' + \kappa_p - (p-k')^2/2M]\tau'} \\
&\quad \quad \varepsilon_s^\mu(k)(2p+k)_\mu \varepsilon_{s'}^\nu(k')(2p' + k')_\nu e^{i[\sigma\kappa - \kappa_p + (p+k)^2/2M]\tau' - i[\sigma\kappa' - \kappa_{p'} + (p+k)^2/2M]\tau} \\
&\quad + \theta(\tau' - \tau) [\varepsilon_s^\mu(k)(2p+k)_\mu \varepsilon_{s'}^\nu(k')(2p' + k')_\nu \\
&\quad \quad e^{i[\sigma\kappa - \kappa_p + (p+k)^2/2M]\tau - i[\sigma\kappa' - \kappa_{p'} + (p+k)^2/2M]\tau'} \\
&\quad \quad \varepsilon_s^\mu(k)(2p' - k)_\mu \varepsilon_{s'}^\nu(k')(2p - k')_\nu e^{i[\sigma\kappa + \kappa_{p'} - (p-k')^2/2M]\tau' - i[\sigma\kappa' + \kappa_p - (p-k')^2/2M]\tau} \} \\
&\hspace{15em} (8.12)
\end{aligned}$$

Treating the integrations on $d\tau$ and $d\tau'$ as in Section 4, we find that

$$\begin{aligned}
\int d\tau d\tau' \theta(\tau - \tau') e^{iA\tau} e^{-iB\tau'} &= \int d\tau e^{iA\tau} \frac{i}{B} e^{-iB\tau} \\
&= (2\pi) \frac{i}{B} \delta(A - B)
\end{aligned} \tag{8.13}$$

where we have taken $B \rightarrow B + i\epsilon$ for the integration. So, (8.12) becomes

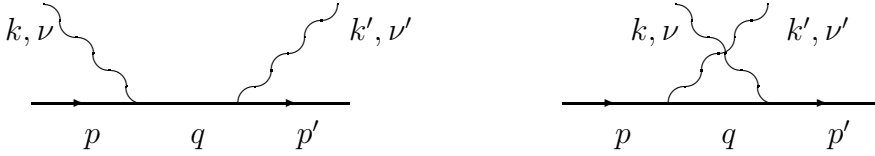
$$\begin{aligned}
&\langle \text{out} | \frac{1}{2!} \int d^5 y d^5 y' \left[\frac{ie_0}{2M} \right]^2 \text{Ta}_\mu(y) j^\mu(y) a_\nu(y') j^\nu(y') | \text{in} \rangle = \\
&= i \frac{g(s)g(s')}{(2\pi)^3 \lambda^2} \left[\frac{e_0}{2M} \right]^2 \delta^4(p+k-p'-k') \delta(\kappa_p - \kappa_{p'} - \sigma\kappa + \sigma\kappa') \\
&\quad \left\{ \frac{\varepsilon_s^\mu(k)(2p' - k)_\mu \varepsilon_{s'}^\nu(k')(2p - k')_\nu}{\frac{1}{2M}(p - k')^2 - (P + \sigma\kappa')} + \frac{\varepsilon_s^\mu(k)(2p+k)_\mu \varepsilon_{s'}^\nu(k')(2p' + k')_\nu}{\frac{1}{2M}(p+k)^2 - (P - \sigma\kappa')} \right\}. \\
&\hspace{15em} (8.14)
\end{aligned}$$

The first term in (8.2) is

$$\begin{aligned}
&\langle \text{out} | i \int d^5 y \frac{e_0^2}{2M} a_\mu(y) a^\mu(y) \psi(y) \psi^*(y) | \text{in} \rangle = \\
&= i \frac{e_0^2}{2M} \int d^5 y \langle 0 | a(k's') a_\mu(y) a^\mu(y) a^*(k, s) | 0 \rangle \langle 0 | b(p') \bar{\phi}(y) \phi(y) b^*(p) | 0 \rangle \\
&= i \frac{e_0^2}{M} \int d^5 y \langle 0 | a(k's') a_\mu(y) | 0 \rangle \langle 0 | a^\mu(y) a^*(k, s) | 0 \rangle \langle 0 | b(p') \bar{\phi}(y) | 0 \rangle \langle 0 | \phi(y) b^*(p) | 0 \rangle \\
&= i \frac{e_0^2}{M} \sum_{r, r'} \int d^5 y \frac{d^4 q}{2\kappa_q} \frac{d^4 q'}{2\kappa_{q'}} \frac{d^4 l}{(2\pi)^4} \frac{d^4 l'}{(2\pi)^4} \varepsilon_r^\mu(q) \varepsilon_{\mu r'}(q') \\
&\quad \langle 0 | a(k's') [a(q, r) e^{i(q \cdot y + \sigma\kappa_q \tau)} + a^*(q, r) e^{-i(q \cdot y + \sigma\kappa_q \tau)}] | 0 \rangle \\
&\quad \langle 0 | [a(q', r') e^{i(q' \cdot y + \sigma\kappa_{q'} \tau)} + a^*(q', r') e^{-i(q' \cdot y + \sigma\kappa_{q'} \tau)}] a^*(k, s) | 0 \rangle
\end{aligned}$$

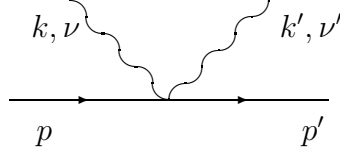
$$\begin{aligned}
& \langle 0|b(p')b^*(l)e^{-i(l \cdot y - \kappa_l \tau)}|0\rangle \langle 0|b(l')e^{i(l' \cdot y - \kappa_{l'} \tau)}b^*(p)|0\rangle \\
&= i \frac{e_0^2}{M} \sum_{r,r'} \int \frac{d^4 q}{2\kappa_q} \frac{d^4 q'}{2\kappa_{q'}} \frac{d^4 l}{(2\pi)^4} \frac{d^4 l'}{(2\pi)^4} \varepsilon_r^\mu(q) \varepsilon_{\mu r'}(q') \\
&\quad (2\pi)^5 \delta^4(q - q' + l - l') \delta(\sigma \kappa_l - \sigma \kappa_{l'} - \kappa_q + \kappa_{q'}) \\
&\quad \frac{2\kappa g(s)}{(2\pi)^4 \lambda} \delta^4(q' - k) \delta_{sr'} \frac{2\kappa' g(s')}{(2\pi)^4 \lambda} \delta^4(q - k') \delta_{s'r} (2\pi)^4 \delta^4(p' - l) (2\pi)^4 \delta^4(p - l') \\
&= i \frac{e_0^2}{M} \frac{g(s)g(s')}{(2\pi)^3 \lambda^2} \varepsilon_s^\mu(k) \varepsilon_{\mu s'}(k') \delta^4(k - k' + p - p') \delta(\sigma \kappa_k - \sigma \kappa_{k'} - \kappa_p + \kappa_{p'}) \quad (8.15)
\end{aligned}$$

By following the scattering matrix Feynman rules of Section 6, we obtain for the first two diagrams



$$\begin{aligned}
& \int d^5 q (1) \times \left[\frac{-ig(s)}{(2\pi)^4 \lambda} \varepsilon_s^\mu(k) \right] \times \left[\frac{ie_0}{2M} (p+q)_\mu (2\pi)^5 \delta^4(k+p-q) \delta(\sigma \kappa_k - \kappa_p + \kappa_q) \right] \\
&\quad \times \left[\frac{-i}{\frac{1}{2M} q^2 - \kappa_p - i\epsilon} \right] \\
&\quad \times \left[\frac{ie_0}{2M} (p'+q)_\nu (2\pi)^5 \delta^4(k'+p'-q) \delta(\sigma \kappa_{k'} - \kappa_{p'} + \kappa_q) \right] \\
&\quad \times \left[\frac{ig(s')}{(2\pi)^4 \lambda} \varepsilon_{s'}^\nu(k') \right] \times (1) + (k, s) \leftrightarrow (k', s') \\
&= i \frac{g(s)g(s')}{(2\pi)^3 \lambda^2} \left[\frac{e_0}{2M} \right]^2 \delta^4(p+k-p'-k') \delta(\kappa_p - \kappa_{p'} - \sigma \kappa + \sigma \kappa') \\
&\quad \left\{ \frac{\varepsilon_s^\mu(k) (2p' - k)_\mu \varepsilon_{s'}^\nu(k') (2p - k')_\nu}{\frac{1}{2M} (p - k')^2 - (P + \sigma \kappa')} + \frac{\varepsilon_s^\mu(k) (2p + k)_\mu \varepsilon_{s'}^\nu(k') (2p' + k')_\nu}{\frac{1}{2M} (p + k)^2 - (P - \sigma \kappa')} \right\} \quad (8.16)
\end{aligned}$$

which we see is identical to (8.14). Similarly, the second diagram



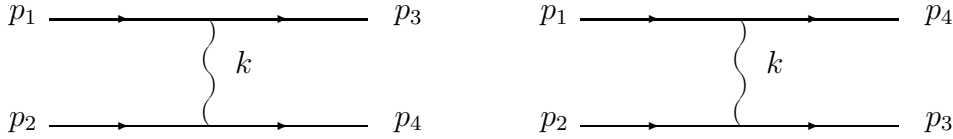
contributes

$$\begin{aligned}
(1) & \times \left[\frac{-ig(s)}{(2\pi)^4 \lambda} \varepsilon_s^\mu(k) \right] \times \left[\frac{-ie_0^2}{M} g_{\mu\nu} (2\pi)^5 \delta^4(k + p - p' - k') \delta(\sigma\kappa_k - \kappa_p - \sigma\kappa_{k'} + \kappa_{p'}) \right] \times \\
& \times \left[\frac{ig(s')}{(2\pi)^4 \lambda} \varepsilon_{s'}^\nu(k') \right] \times (1) + (k, s) \leftrightarrow (k', s') \\
& = i \frac{e_0^2}{M} \frac{g(s)g(s')}{(2\pi)^3 \lambda^2} \varepsilon_s^\mu(k) \varepsilon_{\mu s'}^\nu(k') \delta^4(k - k' + p - p') \delta(\sigma\kappa_k - \sigma\kappa_{k'} - \kappa_p + \kappa_{p'}) \quad (8.17)
\end{aligned}$$

which is identical to (8.15).

Møller Scattering

We now consider the scattering of two identical scalar particles, to first order. The diagrams which contribute are



which contribute

$$\begin{aligned}
\langle 3 \ 4 | iT | 1 \ 2 \rangle & = \\
& = \int d^4 q d\kappa_q (1)^4 \left[\frac{ie_0}{2M} (p_1 + p_3)^\mu (2\pi)^5 \delta^4(p_1 + q - p_3) \delta(\kappa_{p_1} - \kappa_{p_3} - \sigma\kappa_q) \right] \\
& \quad \left[\frac{ie_0}{2M} (p_2 + p_4)^\nu (2\pi)^5 \delta^4(p_2 - q - p_4) \delta(\kappa_{p_2} - \kappa_{p_4} + \sigma\kappa_q) \right] \\
& \quad \frac{1}{\lambda} \frac{-i}{q^2 + \sigma\kappa_q^2 - i\epsilon} \mathcal{P}_{\mu\nu} + (3, 4) \leftrightarrow (4, 3) \\
& = \left\{ i(2\pi)^5 \delta^4(p_1 + p_2 - p_3 - p_4) \delta(\kappa_{p_1} + \kappa_{p_2} - \kappa_{p_3} - \kappa_{p_4}) \right\} (2\pi)^5 \frac{e_0 e}{(2M)^2}
\end{aligned}$$

$$\left\{ \frac{(p_1 + p_3)^\mu (p_2 + p_4)^\nu \left[g_{\mu\nu} + \frac{(p_1 - p_3)_\mu (p_2 - p_4)_\nu}{(p_1 - p_3)^2} \right]}{(p_1 - p_3)^2 + \sigma(\kappa_{p_1} - \kappa_{p_3})^2 - i\epsilon} + (3, 4) \leftrightarrow (4, 3) \right\}, \quad (8.18)$$

where $\kappa_p = \frac{p^2}{2M}$. From (7.10) which defines \mathcal{T} , we have

$$\begin{aligned} \langle 3 \ 4 | \mathcal{T} | 1 \ 2 \rangle &= (2\pi)^5 \frac{e_0 e}{(2M)^2} \left\{ \left[(p_1 + p_3) \cdot (p_2 + p_4) + \frac{(p_1^2 - p_3^2)(p_2^2 - p_4^2)}{(p_1 - p_3)^2} \right] \times \right. \\ &\quad \left. \times \frac{1}{(p_1 - p_3)^2 + \sigma(\kappa_{p_1} - \kappa_{p_3})^2 - i\epsilon} + (3, 4) \leftrightarrow (4, 3) \right\}. \end{aligned} \quad (8.19)$$

At this stage it is convenient to introduce the Mandelstam parameters

$$t = -(p_1 - p_3)^2 = -\frac{1}{4}(p - p')^2 \quad u = -(p_1 - p_4)^2 = -\frac{1}{4}(p + p')^2 \quad (8.20)$$

which complement the definition of s in (7.47), and where p, p' refer to the relative coordinates defined in (7.41) and (7.42). As in the on-shell case, we have

$$\begin{aligned} s + t + u &= -[p_1^2 + p_2^2 + p_1^2 + p_3^2 + p_1^2 + p_4^2 + 2(p_2 - p_3 - p_4) \cdot p_1] \\ &= -[p_1^2 + p_2^2 + p_3^2 + p_4^2] \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2. \end{aligned} \quad (8.21)$$

Similarly, since $p_1^2 + p_2^2 = p_3^2 + p_4^2$ is guaranteed by the ‘fifth’ δ -function, but the masses are not in general invariant, we have

$$\begin{aligned} (p_1 + p_2)^2 &= (p_3 + p_4)^2 \Rightarrow p_1 \cdot p_2 = p_3 \cdot p_4 = -\frac{1}{2}(s + p_1^2 + p_2^2) = -\frac{1}{2}(s + p_3^2 + p_4^2) \\ p_1 \cdot p_3 &= \frac{1}{2}(p_1^2 + p_3^2 + t) \neq p_2 \cdot p_4 = \frac{1}{2}(p_2^2 + p_4^2 + t) \\ p_1 \cdot p_4 &= \frac{1}{2}(p_1^2 + p_4^2 + u) \neq p_2 \cdot p_3 = \frac{1}{2}(p_2^2 + p_3^2 + u) \end{aligned} \quad (8.22)$$

Using these relations among the Mandelstam parameters, we may re-write the terms of (8.19) in the form,

$$\begin{aligned} (p_1 + p_3) \cdot (p_2 + p_4) &= p_1 \cdot p_2 + p_1 \cdot p_4 + p_3 \cdot p_2 + p_3 \cdot p_4 \\ &= -\frac{1}{2}(s + p_1^2 + p_2^2) + \frac{1}{2}(p_1^2 + p_4^2 + u) \\ &\quad + \frac{1}{2}(p_2^2 + p_3^2 + u) - \frac{1}{2}(s + p_3^2 + p_4^2) \\ &= -s + u \end{aligned} \quad (8.23)$$

and similarly

$$(p_1 + p_4) \cdot (p_2 + p_3) = -s + t \quad (8.24)$$

so that

$$\langle 3 \ 4 | \mathcal{T} | 1 \ 2 \rangle = (2\pi)^5 \frac{e_0 e}{(2M)^2} \left\{ \frac{s - u - \frac{(p_1^2 - p_3^2)^2}{t}}{t - \sigma(\kappa_{p_1} - \kappa_{p_3})^2} + \frac{s - t - \frac{(p_1^2 - p_4^2)^2}{u}}{u - \sigma(\kappa_{p_1} - \kappa_{p_4})^2} \right\} \quad (8.25)$$

We notice that for the case of on-shell scattering ($m_1^2 = m_3^2$), the scattering amplitude becomes

$$\langle 3 \ 4 | \mathcal{T} | 1 \ 2 \rangle = (2\pi)^5 \frac{e_0 e}{(2M)^2} \left\{ \frac{s - u}{t} + \frac{s - t}{u} \right\} \quad (8.26)$$

which has the form of the amplitude for the usual scattering of identical Klein-Gordon particles [40].

Consider the amplitude (8.25) in the case of scattering of identical particles with $m_1 = m_2 = m$. In this case, in the center of mass frame, $E(p_1) = E(p_2)$, so that

$$p = p_1 - p_2 = (0, 2\vec{p}_1) \quad (8.27)$$

and $|p| = |p'| = \rho = 2|\vec{p}_1|$. In terms of the parameterization (7.51), the β of the incoming system is just zero. The relative momentum of the outgoing system is then

$$p' = p_3 - p_4 = 2|\vec{p}_1|(\sinh \beta, \cosh \beta \hat{n}) \quad (8.28)$$

and so the Mandelstam parameters become

$$\begin{aligned} t &= -\frac{1}{4}(p - p')^2 \\ &= -\frac{1}{4}(p^2 + p'^2 - 2p \cdot p') \\ &= -\frac{1}{4}(2\rho^2 - 2\rho^2 \cosh \beta \cos \theta) \\ &= -2|\vec{p}_1|^2(1 - \cosh \beta \cos \theta). \end{aligned} \quad (8.29)$$

Similarly,

$$u = -\frac{1}{4}(p + p')^2 = -2|\vec{p}_1|^2(1 + \cosh \beta \cos \theta). \quad (8.30)$$

These expressions agree with the usual on-shell expressions for t and u when $\beta = 0$. Since $p_1^2 = p_2^2$, we may write

$$p_1^2 + p_2^2 = 2p_1^2 = p_3^2 + p_4^2 \quad (8.31)$$

so that

$$\begin{aligned}
p_1^2 - p_3^2 &= \frac{1}{2}(p_3^2 + p_4^2) - p_3^2 \\
&= -\frac{1}{2}(p_3^2 - p_4^2) \\
&= -\frac{1}{2}(p_3 - p_4) \cdot (p_3 + p_4) \\
&= -\frac{1}{2}p' \cdot P' \\
&= \frac{1}{4}\sqrt{s}\rho \sinh \beta
\end{aligned} \tag{8.32}$$

where we have used $P' = P$, (7.46) and (7.51). We see again that $\beta = 0$ corresponds to on-shell scattering. The amplitude in (8.25) contains the term

$$\frac{(p_1^2 - p_3^2)^2}{t} = -\frac{\frac{1}{16}s\rho^2 \sinh^2 \beta}{\frac{1}{2}\rho^2(1 - \cosh \beta \cos \theta)} = -\frac{s \sinh^2 \beta}{8(1 - \cosh \beta \cos \theta)} \tag{8.33}$$

and the denominator

$$t - \sigma(\kappa_{p_1} - \kappa_{p_3})^2 = t - \frac{\sigma}{4M^2}(p_1^2 - p_3^2)^2 = -\frac{\rho}{2} \left[(1 - \cosh \beta \cos \theta) + \frac{\sigma s}{32M^2} \sinh^2 \beta \right] \tag{8.34}$$

Therefore while the usual on-shell scattering amplitude has the single forward direction pole (at $\cos \theta = 1 \Rightarrow t = 0$), the off-shell scattering amplitude has two forward direction poles, one at

$$t = 0 \quad \Rightarrow \quad \cos \theta = \frac{1}{\cosh \beta} \tag{8.35}$$

and the second at

$$t - \sigma(\kappa_{p_1} - \kappa_{p_3})^2 = 0 \quad \Rightarrow \quad \cos \theta = \frac{1}{\cosh \beta} \left[1 + \frac{\sigma s}{32M^2} \sinh^2 \beta \right]. \tag{8.36}$$

Thus, the appearance of two close, but distinct, poles in the forward and backward directions in Møller scattering would be a consequence of Δm^2 not strictly vanishing and would provide a signature for off-shell phenomena. From the optical theorem, we see that the finiteness of the transition amplitude at $\theta = 0$ implies finiteness of the total scattering cross section, for a given $\beta > 0$ (the $-i\epsilon$ term in the denominator of the transition amplitude pushes the pole off the real axis, so that the integral over θ may be performed).

9 Renormalization

Unlike conventional relativistic quantum field theories, the off-shell matter field corresponds to an underlying evolution mechanics, and the field undergoes retarded propagation from τ_1

to $\tau_2 > \tau_1$. Therefore, there can be no matter field loops in off-shell QED, since a closed loop requires the field to propagate from τ_1 to τ_2 and then from τ_2 to τ_1 . The absence of matter field loops leads us to expect that the charge e_0 will not be renormalized, and as we demonstrate below, this follows from the absence of wave function renormalization for the photon.

The Vector Ward Identity

The Ward identity in on-shell (Klein-Gordon) scalar quantum electrodynamics expresses the symmetry associated with the conservation of the four-current as a relationship between the vertex function of the 3-particle interaction and the single particle propagators. Since the Ward identity is preserved at all orders of perturbation theory, it leads to the universality of charge renormalization.

In off-shell quantum electrodynamics, the conservation of current is expressed as a vanishing *five-divergence* (1.24). Recalling equation (6.35)

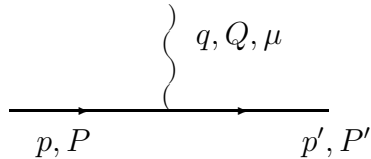
$$\mathcal{L}_{\text{int}} = -\frac{ie_0}{2M}a_\mu(\psi^*\partial^\mu\psi - (\partial^\mu\psi^*)\psi) - \frac{e_0^2}{2M}a_\mu a^\mu|\psi|^2, \quad (9.1)$$

we may express the interaction in terms of the current as

$$\mathcal{L}_{\text{int}} = e_0 a_\mu j^\mu - \frac{e_0^2}{2M} a_\mu a^\mu j^5 \quad (9.2)$$

where (j^μ, j^5) is the current for the free matter field. Since the entire interaction Lagrangian is in the form of *photon* \times *current*, we shall see that the Ward identity is a relationship among the single particle propagators and the vertex functions of *both* the 3-particle interaction and the 4-particle interaction.

We begin with the three-point Green's function associated with the diagram



where $p' = q + p$ and $P' = P - \sigma Q$. To all orders in perturbation theory, the vertex function

(the amputated Green's function) is given by $\Gamma_\mu^{(3)}(p, P; q, Q)$, where

$$G^{(3)\mu}(p, P; q, Q) = G^{(2)}(p, P) G^{(2)}(p', P') d^{\mu\nu}(q, Q) \Gamma_\mu^{(3)}(p, P; q, Q) . \quad (9.3)$$

To calculate the vertex function, we may write [45]

$$\begin{aligned} G_\mu^{(3)}(p, P; q, Q) &= \mathcal{F} \left\{ \langle 0 | T a_\mu(x_1, \tau_1) \psi^*(x_2, \tau_2) \psi(x_3, \tau_3) | 0 \rangle \right\} \\ &= \mathcal{F} \left\{ \langle 0 | T a_\mu(x_1, \tau_1) \psi^*(x_2, \tau_2) \psi(x_3, \tau_3) e^{ie_0 \int d^4x_4 d\tau_4 a_\nu j^\nu} | 0 \rangle_{\text{free}} \right\} \\ &= ie_0 \mathcal{F} \left\{ \int d^4x_4 d\tau_4 \langle 0 | T a_\mu(x_1, \tau_1) a_\nu(x_4, \tau_4) | 0 \rangle_{\text{free}} \right. \\ &\quad \left. \langle 0 | T \psi^*(x_2, \tau_2) \psi(x_3, \tau_3) j^\nu(x_4, \tau_4) | 0 \rangle_{\text{free}} \right\} \end{aligned} \quad (9.4)$$

where \mathcal{F} represents the Fourier transform. The vertex function becomes,

$$\begin{aligned} \Gamma_\mu^{(3)}(p, P; q, Q) &= \frac{1}{G^{(2)}(p)G^{(2)}(p')} \int d^4x d\tau d^4x' d\tau' e^{-i[p \cdot x' - P\tau']} e^{-i[q \cdot x + \sigma Q\tau]} \\ &\quad \langle 0 | T ie_0 j_\mu(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle \end{aligned} \quad (9.5)$$

where we have used translation invariance of the Green's functions to shift one of the field points to zero. Contracting with q^μ , we obtain

$$\begin{aligned} q^\mu \Gamma_\mu^{(3)}(p, P; q, Q) &= e_0 \frac{1}{G^{(2)}(p)G^{(2)}(p')} \int d^4x d\tau d^4x' d\tau' \left[(-\partial_x^\mu) e^{-i[p \cdot x' - P\tau']} e^{-i[q \cdot x + \sigma Q\tau]} \right. \\ &\quad \left. \langle 0 | T j_\mu(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle \right] \\ &= e_0 \frac{1}{G^{(2)}(p)G^{(2)}(p')} \int d^4x d\tau d^4x' d\tau' e^{-i[p \cdot x' - P\tau']} e^{-i[q \cdot x + \sigma Q\tau]} \\ &\quad \langle 0 | T \partial_x^\mu j_\mu(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle , \end{aligned} \quad (9.6)$$

where we have performed one integration by parts. Using current conservation, we may make the replacement

$$\partial_x^\mu j_\mu(x, \tau) = -\partial_\tau j^5(x, \tau) . \quad (9.7)$$

Since the products are τ -ordered, we must carefully differentiate the implied θ -functions to find

$$\begin{aligned} \partial_\tau \langle 0 | T j^5(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle &= \langle 0 | T \partial_\tau j^5(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle \\ &\quad + \delta(\tau - \tau') \langle 0 | T [j^5(x, \tau), \psi(x', \tau')] \psi^*(0) | 0 \rangle \\ &\quad + \delta(\tau) \langle 0 | T [j^5(x, \tau), \psi^*(0)] \psi(x', \tau') | 0 \rangle \end{aligned}$$

$$\begin{aligned}
&= \langle 0 | T \partial_\tau j^5(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle \\
&\quad - \delta(\tau - \tau') \delta^4(x - x') \langle 0 | T \psi(x, \tau) \psi^*(0) | 0 \rangle \\
&\quad + \delta(\tau) \delta^4(x) \langle 0 | T \psi(x', \tau') \psi^*(0) | 0 \rangle
\end{aligned} \tag{9.8}$$

where we have used the commutation relations (4.3). Using (9.7) and (9.8) in (9.6), we find

$$\begin{aligned}
q^\mu \Gamma_\mu^{(3)}(p, P; q, Q) &= e_0 \frac{1}{G^{(2)}(p)G^{(2)}(p')} \int d^4x d\tau d^4x' d\tau' \left[e^{-i[p \cdot x' - P\tau']} e^{-i[q \cdot x + \sigma Q\tau]} \right. \\
&\quad \left[-\partial_\tau \langle 0 | T j^5(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle \right. \\
&\quad \left. - \delta(\tau - \tau') \delta^4(x - x') \langle 0 | T \psi(x, \tau) \psi^*(0) | 0 \rangle \right. \\
&\quad \left. + \delta(\tau) \delta^4(x) \langle 0 | T \psi(x', \tau') \psi^*(0) | 0 \rangle \right] \\
&= e_0 \frac{1}{G^{(2)}(p)G^{(2)}(p')} \int d^4x d\tau d^4x' d\tau' \left[e^{-i[p \cdot x' - P\tau']} e^{-i[q \cdot x + \sigma Q\tau]} \right. \\
&\quad \left[-i\sigma Q \langle 0 | T j^5(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle \right. \\
&\quad \left. - \delta(\tau - \tau') \delta^4(x - x') G^{(2)}(x, \tau) + \delta(\tau) \delta^4(x) G^{(2)}(x', \tau') \right]
\end{aligned} \tag{9.9}$$

where we have performed one integration by parts in the last line. Now, carrying out the integrations, we find that

$$\begin{aligned}
&\int d^4x d\tau d^4x' d\tau' \left[e^{-i[p \cdot x' - P\tau']} e^{-i[q \cdot x + \sigma Q\tau]} \right] \delta(\tau - \tau') \delta^4(x - x') G^{(2)}(x, \tau) \\
&= \int d^4x d\tau e^{-i[(q+p) \cdot x - (P - \sigma Q)\tau]} G^{(2)}(x, \tau) \\
&= G^{(2)}(q + p, P - \sigma Q) \\
&= G^{(2)}(p', P')
\end{aligned} \tag{9.10}$$

and

$$\int d^4x d\tau d^4x' d\tau' \left[e^{-i[p \cdot x' - P\tau']} e^{-i[q \cdot x + \sigma Q\tau]} \right] \delta(\tau) \delta^4(x) G^{(2)}(x', \tau') = G^{(2)}(p, P). \tag{9.11}$$

We notice that the remaining Green's function is proportional to the vertex function for the 4-particle interaction,

$$\begin{aligned}
\Gamma^{(4)}(p, P; q, Q) &= \frac{1}{G^{(2)}(p)G^{(2)}(p')} \int d^4x d\tau d^4x' d\tau' e^{-i[p \cdot x' - P\tau']} e^{-i[q \cdot x + \sigma Q\tau]} \\
&\quad \langle 0 | T \left(-i \frac{e_0^2}{2M} j^5(x, \tau) \right) \psi(x', \tau') \psi^*(0) | 0 \rangle.
\end{aligned} \tag{9.12}$$

Thus, we find

$$\begin{aligned}
q^\mu \Gamma_\mu^{(3)}(p, P; q, Q) &= e_0 \frac{1}{G^{(2)}(p)G^{(2)}(p')} \int d^4x d\tau d^4x' d\tau' \left[e^{-i[p \cdot x' - P \tau']} e^{-i[q \cdot x + \sigma Q \tau]} \right] \\
&\quad (-i\sigma Q) \langle 0 | T j^5(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle \\
&\quad + e_0 \frac{1}{G^{(2)}(p', P')} - e_0 \frac{1}{G^{(2)}(p, P)} \\
&= e_0 (-i\sigma Q) \left(-\frac{2M}{ie_0^2} \right) \Gamma^{(4)}(p, P; q, Q, k, k') \\
&\quad + e_0 \frac{1}{G^{(2)}(p', P')} - e_0 \frac{1}{G^{(2)}(p, P)} \\
&= \sigma Q \frac{1}{e_0/(2M)} \Gamma^{(4)}(p, P; q, Q, k, k') \\
&\quad + e_0 \frac{1}{G^{(2)}(p', P')} - e_0 \frac{1}{G^{(2)}(p, P)} \tag{9.13}
\end{aligned}$$

and the Ward identity takes the form,

$$e_0 q^\mu \Gamma_\mu^{(3)}(p, P; q, Q) - \sigma Q 2M \Gamma^{(4)}(p, P; q, Q, k, k') = e_0^2 \left[\frac{1}{G^{(2)}(p', P')} - \frac{1}{G^{(2)}(p, P)} \right]. \tag{9.14}$$

As expected, in the case that $Q = 0$, (9.14) reduces to the Ward identity for on-shell (Klein-Gordon) scalar QED [45]. At the tree level, where

$$\Gamma^{(4)}(p, P; q, Q, k, k') = \frac{-ie_0^2}{2M} \quad G^{(2)}(p, P) = \frac{-i}{\frac{p^2}{2M} - P} \quad \Gamma_\mu^{(3)}(p, P; q, Q) = i \frac{e_0}{2M} (p + p')_\mu \tag{9.15}$$

we verify that (9.14) is satisfied:

$$\begin{aligned}
e_0 (p' - p)^\mu \cdot \frac{ie_0}{2M} (p' + p)_\mu - \sigma Q M \cdot \frac{-ie_0^2}{M} - e_0^2 \left[i \left(\frac{p'^2}{2M} - P' \right) - i \left(\frac{p^2}{2M} - P \right) \right] = \\
i \frac{e_0^2}{2M} (p'^2 - p^2) + ie_0^2 (P - P') - e_0^2 \left[i \left(\frac{p'^2}{2M} - P' \right) - i \left(\frac{p^2}{2M} - P \right) \right] = 0 \tag{9.16}
\end{aligned}$$

Since the invariance of the Lagrangian is not changed by multiplying each invariant term by a constant, the most general gauge invariant form of the Lagrangian of (2.1), written in terms of renormalized quantities is given by

$$\begin{aligned}
\mathcal{L} &= Z_2 \psi^* \left(i\partial_\tau + \left(\frac{Z_1}{Z_2} \right) e_0 a_5 \right) \psi \\
&\quad - \frac{1}{2Z_4 M} Z_2 \psi^* \left(-i\partial_\mu - \left(\frac{Z_1}{Z_2} \right) e_0 a_\mu \right) \left(-i\partial^\mu - \left(\frac{Z_1}{Z_2} \right) e_0 a^\mu \right) \psi \\
&\quad - \frac{\lambda}{4} \left[Z_3 f_{\mu\nu} f^{\mu\nu} + 2Z_5 f_{5\nu} f^{5\nu} \right] \tag{9.17}
\end{aligned}$$

where we have written the bare field operators in terms of the renormalized field operators as

$$f_B^{\mu\nu} = Z_3 f^{\mu\nu} \quad a_B^\mu = Z_3^{1/2} a^\mu \quad f_B^{5\nu} = Z_5 f^{5\nu} \quad a_B^5 = Z_5^{1/2} a^5 \quad (9.18)$$

$$\psi_B = Z_2^{1/2} \psi . \quad (9.19)$$

Any renormalization of the coupling λ may be absorbed into the wave function renormalizations Z_3 and Z_5 . Consistency requires that

$$\left(\frac{Z_1}{Z_2}\right) e_0 a_5 = \left(\frac{Z_1}{Z_2}\right) e_0 Z_5^{1/2} a_5 \equiv e_0^B a_5^B \quad \text{and} \quad \left(\frac{Z_1}{Z_2}\right) e_0 a_\mu = \left(\frac{Z_1}{Z_2}\right) e_0 Z_3^{1/2} a_\mu \equiv e_0^B a_\mu^B \quad (9.20)$$

so we must have

$$Z_5 = Z_3 \quad e_0 = \frac{Z_2 Z_3^{1/2}}{Z_1} e_0^B . \quad (9.21)$$

We may write the bare Green's functions in terms of the renormalized Green's functions as

$$\begin{aligned} \Gamma_{\mu B}^{(3)} &= \frac{1}{\langle 0|T a_B a_B|0\rangle} \frac{1}{\langle 0|T \psi_B \psi_B^*|0\rangle} \frac{1}{\langle 0|T \psi_B \psi_B^*|0\rangle} \langle 0|T a_B \psi_B \psi_B^*|0\rangle \\ &= \frac{1}{Z_3 \langle 0|T a a|0\rangle} \frac{1}{Z_2 \langle 0|T \psi \psi^*|0\rangle} \frac{1}{Z_2 \langle 0|T \psi \psi^*|0\rangle} Z_3^{1/2} Z_2 \langle 0|T a \psi \psi^*|0\rangle \\ &= \frac{1}{Z_3^{1/2} Z_2} \Gamma_\mu^{(3)} \end{aligned} \quad (9.22)$$

and similarly

$$\Gamma_B^{(4)} = \frac{1}{Z_3 Z_2} \Gamma^{(4)} . \quad (9.23)$$

Since the Ward identity must be valid for the renormalized quantities as well as the unrenormalized quantities, we may compare

$$\begin{aligned} e_0 q^\mu \Gamma_\mu^{(3)}(p, P; q, Q) &- \sigma Q 2M \Gamma^{(4)}(p, P; q, Q, k, k') = \\ &e_0^2 \left[\frac{1}{G^{(2)}(p', P')} - \frac{1}{G^{(2)}(p, P)} \right] \\ \frac{Z_1 Z_3^{1/2}}{Z_2} e_0^B q^\mu Z_3^{1/2} Z_2 \Gamma_{\mu B}^{(3)}(p, P; q, Q) &- \sigma Q 2Z_4 M_B Z_2 Z_3 \Gamma_B^{(4)}(p, P; q, Q, k, k') = \\ &\left(\frac{Z_1 Z_3^{1/2}}{Z_2} e_0^B\right)^2 Z_2 \left[\frac{1}{G_B^{(2)}(p', P')} - \frac{1}{G^{(2)}(p, P)_B} \right] \\ Z_1 Z_3 e_0^B q^\mu \Gamma_{\mu B}^{(3)}(p, P; q, Q) &- Z_2 Z_3 Z_4 \sigma Q 2M_B \Gamma_B^{(4)}(p, P; q, Q, k, k') = \\ &\frac{Z_1^2 Z_3}{Z_2} (e_0^B)^2 \left[\frac{1}{G_B^{(2)}(p', P')} - \frac{1}{G^{(2)}(p, P)_B} \right] \end{aligned}$$

$$e_0^B q^\mu \Gamma_{\mu B}^{(3)}(p, P; q, Q) - \frac{Z_2 Z_4}{Z_1} \sigma Q 2M_B \Gamma_B^{(4)}(p, P; q, Q, k, k') = \frac{Z_1}{Z_2} (e_0^B)^2 \left[\frac{1}{G_B^{(2)}(p', P')} - \frac{1}{G^{(2)}(p, P)_B} \right] \quad (9.24)$$

and find that

$$Z_1 \equiv Z_2 \quad Z_4 \equiv 1 . \quad (9.25)$$

Notice that although the charge e_0 appears linearly and quadratically in the Ward identity, (9.24) makes no restriction on Z_3 , which determines the charge renormalization (by (9.21) with $Z_1 = Z_2$). Thus, the appearance of $\Gamma^{(4)}$ in the Ward identity does not change the universality of charge renormalization found in on-shell QED since the conserved five-current is a consequence of gauge invariance. Nevertheless, since there are no matter field loops in off-shell QED, there are no possible contributions to photon renormalization, and we take $Z_3 \equiv 1$. Therefore, the charge e_0 is not renormalized.

Renormalizability

The remaining renormalization factor is Z_2 , which derives from the renormalization of the matter field by photon loops. These photon loops (matter field self-energy diagrams) will also contribute to the mass renormalization of the matter field, however, the mass term $(\psi^* i \partial_\tau \psi)$ will absorb these contributions. Nevertheless, by examining the primitive divergent self-energy diagrams of successively higher order, it may be seen that in order to make the theory counter-term renormalizable, a cut-off must be applied to the integrations over mass in loop diagrams. For example, at second order, the self-energy diagram with two overlapping photon loops is given by

$$\begin{aligned}
G_2^{(2)}(p) = & G_0^{(2)}(p) G_0^{(2)}(p) \int d^4 q dQ d^4 q' dQ' \left((2\pi)^5 \frac{ie_0}{2M} \right)^4 (2p - q)_\mu (2p - q - 2q')_\nu d^{\mu\nu}(q) \\
& (2p - 2q - q')_\lambda (2p - q')_\sigma d^{\lambda\sigma}(q') \\
& G_0^{(2)}(p - q) G_0^{(2)}(p - q') G_0^{(2)}(p - q - q') .
\end{aligned} \tag{9.26}$$

This expression contains the following term proportional to p^4

$$\begin{aligned}
& G_0^{(2)}(p) G_0^{(2)}(p) \left((2\pi)^5 \frac{ie_0}{2M} \right)^4 16p^4 \int d^4 q dQ d^4 q' dQ' \\
& \frac{1}{\lambda^2} \frac{1}{q^2 + \sigma Q^2} \frac{1}{q'^2 + \sigma Q'^2} \frac{2M}{(p - q)^2 - 2M(P + \sigma Q)} \\
& \frac{2M}{(p - q')^2 - 2M(P + \sigma Q')} \frac{2M}{(p - q - q')^2 - 2M[P + \sigma(Q + Q')]}
\end{aligned} \tag{9.27}$$

and if the mass integrations on dQ and dQ' are taken to infinity, then this term diverges. Since the term is proportional to p^4 , it may not be renormalized by a counter term of a form which appears in the original Lagrangian. However, if the mass integrations are cut off at a finite limit, then the integral is seen to converge. Moreover, the set of divergent diagrams in the resulting theory is just the subset of the divergent diagrams in the on-shell theory which contain no matter field loops. Therefore, with the mass integrations made finite, the off-shell theory is seen to be renormalizable by the same arguments given by Rohrlich [44] for the on-shell theory. Notice that unlike a momentum cut-off, the mass cut-off does not affect the invariances of the original theory. The presense of a finite mass cut-off has a natural interpretation in terms of the correlation limit found by Shnerb and Horwitz [25]. Frastai and Horwitz [26] showed that in the limit of zero-mass photons, the off-shell theory contains a natural regularization of the highest order singularities by an effective Pauli-Villars mass integration [27].

Finally, the one-loop renormalization of the Green's function by two double-photon vertices must also be mentioned. This diagram contributes the logarithmic divergence,

$$\begin{aligned}
G_2^{(4)}(p_1, p_2, p_3, p_4) = & G_0^{(2)}(p_1) G_0^{(2)}(p_2) G_0^{(2)}(p_3) G_0^{(2)}(p_4) \\
& 2 \int d^4 q dQ \left(\frac{-ie_0^2}{M} (2\pi)^5 \right)^2 d_0^{\mu\nu}(q) d_{\mu\nu}^0(p_2 - p_1 + q)
\end{aligned}
\tag{9.28}$$

and as discussed by Rohrlich [44] for on-shell scalar QED, renormalization by counter term requires the introduction of a direct 4-point interaction term for the matter field. However, unlike the usual ϕ^4 theories, this interaction does not introduce matter field loops, because of the retarded propagation in τ . It may be checked that the 6-point and higher diagrams are finite. Since only the photon loops must be considered, the primitive divergent diagrams of the theory contribute only to the matter field self-energy (mass and wavefunction renormalization) and the 4-point interaction term for the matter field.

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